

# SUBSUMPTION ALGORITHMS FOR THREE-VALUED GEOMETRIC RESOLUTION

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**ABSTRACT.** In our implementation of geometric resolution, the most costly operation is subsumption testing (or matching): One has to decide for a three-valued, geometric formula, if this formula is false in a given interpretation. The formula contains only atoms with variables, equality, and existential quantifiers. The interpretation contains only atoms with constants. Because the atoms have no term structure, matching for geometric resolution is hard. We translate the matching problem into a generalized constraint satisfaction problem, and discuss several approaches for solving it, two direct algorithms and two translations to propositional SAT. After that, we study filtering techniques based on local consistency checking. Such filtering techniques can a priori refute a large percentage of generalized constraint satisfaction problems. Finally, we adapt the matching algorithms in such a way that they find solutions that use a minimal subset of the interpretation. The adaptation can be combined with every matching algorithm. The techniques presented in this paper may have applications in constraint solving.

## 1. INTRODUCTION

Geometric logic as a theorem proving strategy was introduced in [1]. (The infinitary variant is called *coherent logic*.) Bezem and Coquand were motivated mostly by the desire to obtain a theorem proving strategy with a simple normal form transformation, which makes that many natural problems need no transformation at all, others have a much simpler transformation, and which makes that in all cases Skolemization can be avoided. This results in more readable proofs, and proofs that can be backtranslated more easily.

Our motivation for using geometric resolution is different, more engineering-oriented. We hope that three-valued, geometric resolution can be used as a generic reasoning core, into which different kinds of two- or three-valued reasoning or decision problems (e.g. problems representing type correctness, two-valued decision problems, simply typed classical problems) can be solved. Because we want the geometric reasoning core to be generic, we are willing to accept transformations that do not preserve much of the structure of the original formula. Subformulas are freely renamed, and functional expressions are flattened and replaced by relations.

We start by giving a definition of three-valued, geometric formulas. The definition that we give here is slightly too general, but easier to understand than the correct definition in [3], which contains some additional, technical restrictions that are required by other parts of the search algorithm.

**Definition 1.1.** A *geometric literal* has one of the following four forms:

- (1) A simple atom of form  $p_\lambda(x_1, \dots, x_n)$ , where  $x_1, \dots, x_n$  are variables (with repetitions allowed) and  $\lambda \in \{\mathbf{f}, \mathbf{e}, \mathbf{t}\}$ . (denoting *false*, *error* and *true*.)
- (2) An equality atom of form  $x_1 \approx x_2$ , with  $x_1, x_2$  distinct variables.
- (3) A domain atom  $\#_{\mathbf{f}} x$ , with  $x$  a variable.
- (4) An *existential atom* of form  $\exists y p_\lambda(x_1, \dots, x_n, y)$  with  $\lambda \in \{\mathbf{f}, \mathbf{e}, \mathbf{t}\}$ , and such that  $y$  occurs at least once in the atom, not necessarily on the last place.

A *geometric formula* has form  $A_1, \dots, A_p \mid B_1, \dots, B_q$ , where the  $A_i$  are simple or domain atoms, and the  $B_j$  are atoms of arbitrary type.

We require that geometric formulas are *range restricted*, which means that every variable that occurs free in a  $B_j$  must occur in an  $A_i$  as well.

The intuitive meaning of  $A_1, \dots, A_p \mid B_1, \dots, B_q$  is  $\forall \bar{x} A_1 \vee \dots \vee A_p \vee B_1 \vee \dots \vee B_q$ , where  $\bar{x}$  are all the free variables. The vertical bar ( $\mid$ ) has no logical meaning. Its only purpose is to separate the two types of atoms.

A geometric formula that is not range restricted, can always be made range restricted by inserting suitable  $\#_{\mathbf{f}}$  atoms into the left hand side. This is the only purpose of the  $\#$ -predicate. Interpretations contain predicates of form  $\#_{\mathbf{t}} c$ , for every domain element  $c$ . Atoms in geometric formulas are variable-only, and are labeled with truth-values, as in [13]. It is shown in [4] and [3] that formulas in classical logic with partial functions ([2]) can be translated into sets of geometric formulas.

**Definition 1.2.** We define an *interpretation*  $I$  as a finite set of atoms of forms  $\# c$  with  $c$  a constant, or form  $p_\lambda(x_1, \dots, x_n)$ , where  $x_1, \dots, x_n$  are constants (repetitions allowed). Interpretations must be *range restricted* as well. This means that every constant  $x$  occurring in the interpretation must occur in an atom of form  $\#_{\mathbf{t}} x$ .

Matching searches for false formulas. These are formulas whose premises  $A_1, \dots, A_p$  clash with  $I$ , while none of the  $B_j$  is true in  $I$ .

**Definition 1.3.** Let  $I$  be an interpretation. Let  $A$  be a geometric literal. Let  $\Theta$  be a substitution that assigns constants to variables, and that is defined on the variables in  $A$ . We say that  $A\Theta$  *conflicts* (or *is in conflict with*)  $I$  if **(1)**  $A$  has form  $p_\lambda(x_1, \dots, x_n)$ , and there is an atom of form  $p_\mu(x_1\Theta, \dots, x_n\Theta) \in I$  with  $\lambda \neq \mu$ , **(2)**  $A$  has form  $x_1 \approx x_2$  and  $x_1\Theta \neq x_2\Theta$ , or **(3)**  $A$  has form  $\#_{\mathbf{f}} x$  and  $(\#_{\mathbf{t}} x\Theta) \in I$ .

We say that  $A\Theta$  *is true* in  $I$  if

- (1)**  $A$  has form  $p_\lambda(x_1, \dots, x_n)$  and  $p_\lambda(x_1\Theta, \dots, x_n\Theta) \in I$ ,
- (2)**  $A$  has form  $x_1 \approx x_2$  and  $x_1\Theta = x_2\Theta$ ,
- (3)**  $A$  has form  $\#_{\mathbf{t}} x$  and  $(\#_{\mathbf{t}} x\Theta) \in I$ , or
- (4)**  $A$  has form  $\exists y B_\lambda(x_1, \dots, x_n, y)$  and there exists a constant  $c$ , s.t.  $B_\lambda(x_1\Theta, \dots, x_n\Theta, c) \in I$ .

In the definitions of truth and conflict,  $\#$  is treated as a usual predicate.

**Definition 1.4.** Let  $I$  be an interpretation. Let  $B$  be a geometric atom. Let  $\Theta$  be a substitution that instantiates all free variables of  $B$ , and for which  $B\Theta$  is not true in  $I$ . We define the *extension set*  $E(B, \Theta)$  as follows:

- If  $B$  has form  $p_\lambda(x_1, \dots, x_n)$  or  $\#_{\mathbf{t}} x$ , then  $E(B, \Theta) = \{B\Theta\}$ .
- If  $B$  has form  $x_1 \approx x_2$ , then  $E(B, \Theta) = \emptyset$ .
- If  $B$  has form  $\exists y B_\lambda(x_1, \dots, x_n, y)$ , then

$$E(B, \Theta) = \{ B\Theta\{y := c\} \mid c \in I \} \cup \{ B\Theta\{y := \hat{c}\} \}.$$

By  $c \in I$  we mean:  $c$  is a constant occurring in an atom of  $I$ . We assume that  $\hat{c}$  is the first constant for which  $\hat{c} \notin I$ .

Intuitively, if for a geometric formula  $\phi = A_1, \dots, A_p \mid B_1, \dots, B_q$  and a substitution  $\Theta$ , the  $A_i\Theta$  are in conflict with  $I$ , while none of the  $B_j\Theta$  is true in  $I$ , then  $\phi\Theta$  is false in  $I$ . If there exist a  $B_j$  and an atom  $C \in E(B_j, \Theta)$  that is not in conflict with  $I$ , then  $\phi\Theta$  can be made true by adding  $C$ . If no such  $C$  exists, a conflict was found. If more than one  $C$  exists, the search algorithm has to backtrack through all possibilities. The search algorithm tries to extend an initial interpretation  $I$  into an interpretation  $I' \supset I$  that makes all formulas true. At each stage of the search, it looks for a formula and a substitution that make the formula false. If no formula and substitution can be found, the current interpretation is a model. Otherwise, search continues either by extending  $I$ , or by backtracking. Details of the procedure are described in [6] for the two-valued case, and in [3] for the three-valued case. Experiments with the current three-valued version (available from [15]), and the previous two-valued version ([7]) show that the search for false formulas consumes nearly all of the resources of the prover.

**Definition 1.5.** An instance of *the matching problem* consists of an interpretation  $I$  and a geometric formula  $A_1, \dots, A_p \mid B_1, \dots, B_q$ .

Determine if there exists a substitution  $\Theta$  that brings all  $A_i$  in conflict with  $I$ , and makes none of the  $B_j$  true in  $\Theta$ . If yes, then return such substitution.

**Examples 1.6.** Consider an interpretation  $I$  consisting of atoms

$$P_t(x_0, x_0), P_e(x_0, x_1), P_t(x_1, x_1), P_e(x_1, x_2), Q_t(x_2, x_0).$$

The formula  $\phi_1 = P_f(X, Y), P_f(Y, Z) \mid Q_t(Z, X)$  can be matched in five ways:

$$\begin{aligned} \Theta_1 &= \{ X := x_0, Y := x_0, Z := x_0 \} \\ \Theta_2 &= \{ X := x_0, Y := x_0, Z := x_1 \} \\ \Theta_3 &= \{ X := x_0, Y := x_1, Z := x_1 \} \\ \Theta_4 &= \{ X := x_1, Y := x_1, Z := x_1 \} \\ \Theta_5 &= \{ X := x_1, Y := x_1, Z := x_2 \} \end{aligned}$$

The substitution  $\Theta_6 = \{ X := x_0, Y := x_1, Z := x_2 \}$  would make the conclusion  $Q_t(Z, X)$  true. Next consider the formula  $\phi_2 = P_f(X, Y), P_t(Y, Z) \mid X \approx Y$ .

The substitution  $\Theta = \{ X := x_0, Y := x_1, Z := x_2 \}$  is the only matching of  $\phi_2$  into  $I$ . Finally, the formula  $\phi_3 = P_t(X, Y) \mid \exists Z Q_t(Y, Z)$  can be matched with  $\Theta = \{ X := x_0, Y := x_1 \}$ , and in no other way.

The first formula  $\phi_1$  in example 1.6 has five matchings. In case there exists more than one matching, it matters for the geometric prover which matching is returned. This is because the prover analyses which ground atoms in the interpretation  $I$  contributed to the matching, and will consider only those in backtracking. In general, the set of conflicting atoms in  $I$  should be as small as possible, and should depend on as few as possible decisions. (Decisions in the sense of propositional reasoning, see [12].) The simplest solution for finding the best matching would be to enumerate all matchings, and use some preference relation  $\preceq$  to keep the best one. Unfortunately, this approach is not practical because the number of matchings can be extremely high. We will address this problem in Section 9.

Even if one is interested in the decision problem only, matching is still intractable because the decision problem is already NP-complete. This can be shown by a simple reduction from SAT.

In this paper, we introduce several algorithms for efficiently solving the matching problem. The algorithms evolved out of predecessors that have been implemented before in the

two-valued version of **Geo** ([7]), and in the three-valued version of **Geo** that took part in CASC J8 (see [15]). The algorithm of the three-valued version is discussed in detail in [5]. Unfortunately, after comparison with other methods, in particular the algorithms in the current paper, and translation to SAT, the approach of [5] turned out not competitive, and we have abandoned it. On the positive side, we now have matchings algorithms that are on average 500-1000 times faster than the algorithm of [5].

The paper is organized as follows: In Section 2, we translate the matching problem into a structure called *generalized constraint satisfaction problem* (GCSP). The generalization consists of the fact that it contains additional constraints, that a solution must not make true. These constraints correspond to the conclusions of the geometric formula that one is trying to match.

After that, we present in Section 3 a backtracking algorithm for solving GCSP, which is based on backtracking combined with a form of propagation. It relies on a data structure that we call *refinement stack*. Refinements stacks were introduced in [5]. The main matching algorithm turned out non-competitive, but its data structure is still useful. In Section 4 we discuss conflict learning. In Section 5, we briefly discuss the algorithm of [5]. In Section 6, we give two translations from GCSP to SAT. The translations are straightforward, and very efficiently solved by MiniSat ([9]). In Section 7, we present another matching algorithm that is currently not implemented, but which might be competitive. In Section 8, we discuss experimental results. In Section 9, we explain how every algorithm that is able to find some solution, can be transformed into an algorithm that finds an optimal solution, from the point of view of geometric resolution. In Section 10, we present a priori filtering techniques, that are able to reject a priori a large percentage GCSPs.

## 2. TRANSLATION INTO GENERALIZED CONSTRAINT SATISFACTION PROBLEM

We introduce the generalized constraint satisfaction problem, and show how instances of the matching problem can be translated. It is ‘generalized’ because there are additional, negative constraints (called *blockings*), which a solution is not allowed to satisfy. The blockings originate from translations of the  $B_1, \dots, B_q$ .

**Definition 2.1.** A *substlet*  $s$  is a (small) substitution. We usually write  $s$  in the form  $\bar{v}/\bar{x}$ , where  $\bar{v}$  is a sequence of variables without repetitions, and  $\bar{x}$  is a sequence of constants of same length as  $\bar{v}$ .

We say that two substlets  $\bar{v}_1/\bar{x}_1$  and  $\bar{v}_2/\bar{x}_2$  are *in conflict* if there exist  $i, j$  s.t.  $v_{1,i} = v_{2,j}$  and  $x_{1,i} \neq x_{2,j}$ .

If  $\bar{v}_1/\bar{x}_1, \dots, \bar{v}_n/\bar{x}_n$  is a sequence of substlets not containing a conflicting pair, then one can merge them into a substitution as follows:  $\bigcup\{\bar{v}_1/\bar{x}_1, \dots, \bar{v}_n/\bar{x}_n\} = \{v_{i,j} := x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq \|\bar{v}_i\|\}$ .

If  $\Theta$  is a substitution and  $s = \bar{v}/\bar{x}$  is a substlet, we say that  $\Theta$  makes  $s$  *true* if every  $v_i := x_i$  is present in  $\Theta$ .

We say that  $\Theta$  and  $s$  are in conflict if there is a  $v_i/x_i$  with  $1 \leq i \leq \|v\|$ , s.t.  $v_i\Theta$  is defined and distinct from  $x_i$ .

A *clause*  $c$  is a finite set of substlets with the same domain. We say that a substitution  $\Theta$  *makes  $c$  true* (notation  $\Theta \models c$ ) if  $\Theta$  makes a substlet  $(\bar{v}/\bar{x}) \in c$  true. We say that  $\Theta$  *makes  $c$  false* (notation  $\Theta \models \neg c$ ) if every substlet  $(\bar{v}/\bar{x}) \in c$  is in conflict with  $\Theta$ . In the remaining case, we call  $c$  *undecided by  $\Theta$* .

**Definition 2.2.** A *generalized constraint satisfaction problem* (GCSP) is a pair of form  $(\Sigma^+, \Sigma^-)$  in which  $\Sigma^+$  is a finite set of clauses, and  $\Sigma^-$  is a finite set of substlets.

A substitution  $\Theta$  is a *solution* of  $(\Sigma^+, \Sigma^-)$ , if every clause in  $\Sigma^+$  is true in  $\Theta$ , and there is no  $\sigma \in \Sigma^-$ , s.t.  $\Theta$  makes  $\sigma$  true.

**Definition 2.3.** Let  $(\Sigma^+, \Sigma^-)$  a GCSP. We call  $(\Sigma^+, \Sigma^-)$  *range restricted* if for every variable  $v$  that occurs in a substlet  $\sigma \in \Sigma^-$ , there exists a clause  $c \in \Sigma^+$  s.t. every substlet  $s \in c$  has  $v$  in its domain.

We now explain how a matching instance is translated into a generalized constraint satisfaction problem.

**Definition 2.4.** Assume that  $I$  and  $\phi = A_1, \dots, A_p \mid B_1, \dots, B_q$  together form an instance of the matching problem. The *translation*  $(\Sigma^+, \Sigma^-)$  of  $(I, \phi)$  into GCSP is obtained as follows:

- For every  $A_i$ , let  $\bar{v}_i$  denote the variables of  $A_i$ . Then  $\Sigma^+$  contains the clause  $\{ \bar{v}_i / \bar{v}_i \Theta \mid A_i \Theta \text{ is in conflict with } I \}$ .
- For every  $B_j$ , let  $\bar{w}_j$  denote the variables of  $B_j$ . For every  $\Theta$  that makes  $B_j \Theta$  true in  $I$ ,  $\Sigma^-$  contains the substlet  $\bar{w}_j / (\bar{w}_j \Theta)$ .

**Theorem 2.5.** A matching instance  $(I, \phi)$  has a matching iff its corresponding GCSP has a solution.

In theory, the set of blockings  $\Sigma^-$  can be removed, because a blocking  $\sigma$  can always be replaced by a clause as follows: Let  $\sigma$  be a blocking, let  $\bar{v}$  be its variables. Define  $\sigma_1 = \sigma$ , and let  $\sigma_2, \dots, \sigma_n \in \Sigma^-$  be the blockings whose domain is also  $\bar{v}$ . One can replace  $\sigma_1, \dots, \sigma_n$  by the clause  $\{ \bar{v} / \bar{c} \mid \bar{v} / \bar{c} \text{ conflicts all } \sigma_i (1 \leq i \leq n) \}$ .

We prefer to keep  $\Sigma^-$ , because in the worst case, the resulting clause has size  $m^{\|\bar{v}\|}$ , where  $m$  is the size of the domain. For example, if  $\sigma_1, \dots, \sigma_n$  result from an equality  $X \approx Y$ , then  $\sigma_i$  has form  $(X, Y) / (x_i, x_i)$ . The resulting clause  $c = \{ (X, Y) / (x_i, x_j) \mid i \neq j \}$  has size  $n(n-1) \approx n^2$ .

Clauses resulting from a matching problem have the following trivial, but essential property:

**Lemma 2.6.** Let  $(\Sigma^+, \Sigma^-)$  be obtained by the translation in Definition 2.4. Let  $s_1, s_2 \in c \in \Sigma^+$ . Then either  $s_1 = s_2$ , or  $s_1$  and  $s_2$  are in conflict with each other.

Lemma 2.6 holds because  $s_1$  and  $s_2$  have the same domain.

**Examples 2.7.** In example 1.6, the matching problem  $(I, \phi_1)$  can be translated into the GCSP below. The clauses are above the horizontal line, and the blockings are below it. Because substlets in the same clause always have the same variables, we write the variables of a clause only once.

$$\begin{array}{c} (X, Y) / (x_0, x_0) \mid (x_0, x_1) \mid (x_1, x_1) \mid (x_1, x_2) \\ (Y, Z) / (x_0, x_0) \mid (x_0, x_1) \mid (x_1, x_1) \mid (x_1, x_2) \\ \hline (X, Z) / (x_0, x_2) \end{array}$$

Translating  $(I, \phi_2)$  results in:

$$\frac{\begin{array}{l} (X, Y) / (x_0, x_0) \mid (x_0, x_1) \mid (x_1, x_1) \mid (x_1, x_2) \\ (Y, Z) / (x_0, x_1) \mid (x_1, x_2) \end{array}}{\begin{array}{l} (X, Y) / (x_0, x_0) \\ (X, Y) / (x_1, x_1) \\ (X, Y) / (x_2, x_2) \end{array}}$$

Translation of  $(I, \phi_3)$  results in:

$$\frac{(X, Y) / (x_0, x_1) \mid (x_1, x_2)}{(Y) / (x_2)}$$

Before one runs any algorithms on a GCSP, it is useful to do some simplifications. If the GCSP contains a propositional clause (not containing any variables), this clause is either the empty clause, or a tautology. In the first case, the problem is trivially unsolvable. In the second case, the clause can be removed.

Similarly, if  $\Sigma^-$  contains a propositional blocking, then  $(\Sigma^+, \Sigma^-)$  is trivially unsolvable. Such blockings originate from a  $B_j$  that is purely propositional, or that has form  $\exists y P_\lambda(y)$ .

A third important preprocessing step is *removal of unit blockings*. Let  $\sigma \in \Sigma^-$  be a blocking whose domain is included in the domain of some clause  $c \in \Sigma^+$ . In that case, one can remove every substlet  $\bar{v}/\bar{c}$  from  $c$ , that has  $\bigcup\{\bar{v}/\bar{c}\} \models \sigma$ . If this results in  $c$  being empty, then  $(\Sigma^+, \Sigma^-)$  trivially has no solution. If no  $\bar{v}/\bar{c}$  in any clause  $c \in \Sigma^+$  implies  $\sigma$ , then  $\sigma$  can be removed from  $\Sigma^-$ , because of Lemma 2.6.

Applying removal of unit blockings to the translation of  $(I, \phi_2)$  above results in

$$\frac{\begin{array}{l} (X, Y) / (x_0, x_1) \mid (x_1, x_2) \\ (Y, Z) / (x_0, x_1) \mid (x_1, x_2) \end{array}}{\quad}$$

It is worth noting that removal of propositional blockings can be viewed as a special case of removal of unit blockings.

A GCSP can be solved by backtracking, similar to SAT solving. A backtracking algorithm for GCSP can be either variable or clause based. A variable based algorithm maintains a substitution  $\Theta$ , which it tries to extend into a solution. It backtracks by picking a variable  $v$  and trying to assign it in all possible ways. It backtracks when  $\Theta$  makes a clause  $c \in \Sigma^+$  false, or a blocking  $\sigma \in \Sigma^-$  true.

A clause based algorithm maintains a consistent set  $S$  of substlets (whose union defines a substitution). It backtracks by picking an undecided clause  $c \in \Sigma^+$ , and consecutively inserting all substlets that are consistent with  $S$  into  $S$ . It backtracks when there is a clause  $c$  all of whose atoms are in conflict with  $S$ , or when  $\bigcup S$  makes a blocking true.

Our experiments suggest that there is no significant difference in performance, nor in programming effort, between the two variants. We will stick with clause based algorithms, because it seems that they can be more easily combined with local consistency checking.

### 3. MATCHING USING REFINEMENT STACKS

We first present a matching algorithm without learning, and add learning in the next section. The algorithm that we present here is a simplification of the algorithm in [5], which unfortunately could not be made competitive. The previous algorithm was based on a combination of local consistency checking and lemma learning from conflicts. Local

consistency checking will be discussed in detail in Section 10, because there is still a high probability that it can be used as priori check.

Local consistency checking means that one generates all subsets of clauses up to some size  $S + 1$  and checks which substlets can occur in solutions. Substlets that do not occur in any solution of some subset, certainly do not occur in a solution of the complete GCSP. In most instances, filtering with a small  $S$ , e.g. 1 or 2 results in an empty clause. The algorithm of [5] was based on a combination of local consistency checking and decision. It is discussed in more detail in Section 5.

The algorithm that we discuss in this section evolved from [5]. The main differences are: Clauses are not checked against each other anymore. Instead, clauses are checked only against the substitution in combination with blockings. Secondly, learnt lemmas are flat, i.e. finite disjunctions of single assignments to variables. In [5], lemmas were finite disjunctions of substlets. It turns out that this simplification improves performance by a factor between 100 and 1000.

In order to implement matching algorithms and local consistency checking, one needs to be able to remove substlets from clauses, and reintroduce them during backtracking. We call the process of removing substlets from a clause *refinement*. Whenever a clause has been refined, it may trigger other refinements. In the current algorithm, refinement of a clause only triggers possible extension of the substitution. In the local consistency check, refinement of a clause may trigger further refinements in other clauses. As a consequence, one needs to maintain a queue of recent refinements and use this queue to check which more clauses can be refined. We introduce a data structure, called *refinement stack* which supports refinement of clauses, restoring during backtracking, and keeping track of unchecked refinements.

**Definition 3.1.** A *refinement* has form  $c \Rightarrow d$ , where both  $c$  and  $d$  are clauses, and  $d$  is a subclause of  $c$ .

A *refinement stack*  $\overline{C}$  is a finite sequence of refinements  $c_i \Rightarrow d_i$ . If there exists a  $j$  with  $i < j$  and  $c_i = c_j$ , then  $d_j$  must be a strict subclause of  $c_i$ .

For a clause  $c$ , if  $c_i \Rightarrow d_i$  is the last refinement with  $c = c_i$  occurring in  $\overline{C}$ , we call  $d_i$  *the current refinement of  $c$* .

We define a predicate  $\alpha_i(\overline{C})$  that is true if  $c_i \Rightarrow d_i$  is the current refinement of  $c_i$  in  $\overline{C}$ . This means that there is no  $j > i$  with  $c_j = c_i$ .

A refinement stack supports gradual refinement of clauses. If  $\alpha_i(\overline{C})$  is true, then clause  $d_i$  can be refined into  $d'$  by appending  $c_i \Rightarrow d'$  to  $\overline{C}$ .

In the new refinement stack  $\overline{C}' = \overline{C} + (c_i \Rightarrow d')$ , we have  $c_i = c_{n+1}$ ,  $\alpha_i(\overline{C}')$  is false, and  $\alpha_{n+1}(\overline{C}')$  is true.

The *size*  $\|\overline{C}\|$  of a refinement stack  $\overline{C}$  is defined as the total number of refinements that occur in it, independent of the values of  $\alpha_i(\overline{C})$ .

The refinement stack is initialized with the refinements  $c \Rightarrow c$ , for each initial clause  $c$ . Refinement stacks can be efficiently implemented without need to copy clauses by maintaining a stack of intervals of active substlets in the initial clauses. A substlet can be disabled by swapping it with the last active substlet in the interval, and decreasing the size of the interval by one. When the substlet is made active again, it is sufficient to restore the interval, because the order of active substlets in a clause does not matter. Refinement stacks support change driven inspection as well as backtracking. Later, in Definition 7.1, we will

also introduce substitution stacks, which are similar to refinement stacks, but which allow the gradual refinement of substitutions.

Change driven inspection of clauses can be implemented by starting at position  $k = 1$ . As long as  $k \leq \|\overline{C}\|$ , one first checks  $\alpha_k(\overline{C})$ . If it is false, then  $d_k$  is not the current version of  $c_k$ , and one can increase  $k$ . If  $\alpha_k(\overline{C})$  is current, one can check if  $d_k$  triggers refinement of other clauses. If yes, the results are inserted at the end, so that they will be inspected at later time. When one reaches  $k > \|\overline{C}\|$ , one has reached a stable state.

When some change involving a variable  $v$  takes place, one needs to check which clauses may be affected by the change, so that they can be refined. These are obviously the clauses that contain  $v$ , but also the clauses that contain a variable occurring in a blocking that contains  $v$ , since the algorithm takes blockings into account, when refining. This gives rise to the following definition:

**Definition 3.2.** Let  $v, w$  be two variables. We call  $v$  and  $w$  *connected* if  $v$  and  $w$  occur together in a blocking  $\sigma \in \Sigma^-$ .

We define the search algorithm. We assume that propositional clauses and unit blockings have been removed from  $(\Sigma^+, \Sigma^-)$ . We will assume that the substitution  $\Theta$  is an ordered sequence (stack) of assignments  $(v_1/x_1, \dots, v_s/x_s)$ .

**Algorithm 3.3.** We want to find a solution for  $(\Sigma^+, \Sigma^-)$ . Initially, set  $\Theta := \emptyset$  and  $\overline{C} := \emptyset$ . After that, for each  $k$  ( $1 \leq k \leq \|\Sigma^+\|$ ), do the following:

**PREPROC:** Let  $c_k$  be the  $k$ -th clause in  $\Sigma^+$ . Append  $(c_k \Rightarrow c_k)$  to  $\overline{C}$ . For every variable  $v$  occurring in  $c_k$ , for which all substlets in  $c_k$  agree on the value of  $v$ , let  $x$  be the agreed value.

- If  $v\Theta$  is undefined, and there is a blocking  $\sigma$  containing  $v$ , s.t.  $\Theta \cup \{v/x\} \models \sigma$ , then return  $\perp$ . Otherwise, append  $v/x$  to  $\Theta$ .
- If  $v\Theta$  is defined, and  $v\Theta \neq x$ , then return  $\perp$ . Otherwise, do nothing.

After that, we call the main search algorithm **findmatch** $(\overline{C}, \Theta, s, \Sigma^-)$  with  $s = 1$ . It either returns  $\perp$ , or it extends  $\Theta$  into a solution of  $(\overline{C}, \Sigma^-)$ . **findmatch** $(\overline{C}, \Theta, s, \Sigma^-)$  is defined as follows:

**FORW:** As long as  $s \leq \|\Theta\|$ , let  $v/x$  be the  $s$ -th assignment of  $\Theta$ .

- (1) For every  $(c_i \Rightarrow d_i) \in \overline{C}$  which has  $\alpha_i(\overline{C})$  true, and which either contains  $v$  itself, or a variable  $w$  that is connected to  $v$ , let

$$d' = \{s \in d_i \mid s \text{ is not in conflict with } \Theta\}.$$

If  $d' = \emptyset$ , then return  $\perp$ . Otherwise, let

$$d'' = \{s \in d' \mid \text{there is no } \sigma \in \Sigma^-, \text{ s.t. } \Theta \cup \{s\} \models \sigma\}.$$

If  $d'' = \emptyset$ , then return  $\perp$ . Otherwise, if  $d'' \subset d_i$ , then

- (a) append  $(c_i \Rightarrow d'')$  to  $\overline{C}$ .
- (b) For every variable  $v'$  occurring in  $d''$ , that is unassigned in  $\Theta$ , for which all substlets in  $d''$  agree on the assigned value, let  $x'$  be the agreed value. Append  $v'/x'$  to  $\Theta$ .

- (2) Set  $s = s + 1$ .

**PICK:** Find an  $i$  with  $\alpha_i(\overline{C})$  true and  $\|d_i\| > 1$ . If no such  $i$  exists, then  $\Theta$  is a solution.

Otherwise, for every substlet  $\overline{v}_j/\overline{x}_j$  in  $d_i$ , do the following:



- (1) Append  $c_i \Rightarrow (\bar{v}_j/\bar{x}_j)$  to  $\bar{C}$ , and extend  $\Theta$  with the unassigned variables in  $\bar{v}_j/\bar{x}_j$ .
  - (2) Recursively call **findmatch**(  $\bar{C}, \Theta, s, \Sigma^-$  ). If  $\Theta$  was extended into a solution, then return  $\Theta$ .
  - (3) Otherwise, restore  $\Theta$  and  $\bar{C}$  to the sizes that they had before (1).
- At this point, each of the recursive calls has returned  $\perp$ . Return  $\perp$ .

Algorithm 3.3 is similar to DPLL in that it tries to postpone backtracking as long as possible by giving preference to deterministic extension. Deterministic extension is attempted at **FORW**. If no further deterministic extension is possible then a non-unit clause is replaced by one of its subtllets at **PICK**, after which further deterministic extension is attempted. At **FORW**, blockings are taken into account. It is possible to implement **FORW** without considering blockings. In that case, it has to be checked, whenever the substitution is extended (at **PICK 2** and at **FORW 1b**) that the extended substitution does not imply a blocking. The given version performs better in experiments.

Algorithm 3.3 can be implemented without refinement stack, but that would cause a lot of rechecking, because a clause typically contains many variables and may be connected to even more, so that it would have to be rechecked every time when one of its variables becomes instantiated.

In order to show that Algorithm 3.3 is correct, i.e. does not report false solutions, we have to show that all necessary checks are made.

**Lemma 3.4.** (1) *At points **FORW** and **PICK** of Algorithm 3.3, there is no  $\sigma \in \Sigma^-$ , s.t.  $\Theta \models \sigma$ .*  
 (2) *At point **PICK**, no refined clause  $d_i$  contains a subtllet that is in conflict with  $\Theta$ .*

Initially, the preprocessor ensures that there is no  $\sigma \in \Sigma^-$ , s.t.  $\Theta \models \sigma$ . When  $\sigma$  is extended in **FORW 1b**, it has been checked before that  $\Theta \cup \{s\}$  does not imply a blocking, for each of the subtllets in  $d''$ .

At point **PICK**, **findmatch** passed through **FORW** which refined away all subtllets that conflict with  $\Theta$ .

In the next section, we will extend Algorithm 3.3 with learning. This will prove completeness, because whenever Algorithm 3.3 does not find a solution, it will construct a lemma that proves that no lemma exists.

#### 4. CONFLICT LEARNING

It is known from propositional SAT solving that conflict learning dramatically improves the performance of SAT solvers ([12]). The matching algorithm in the two-valued version of **Geo** ([7]) was already equipped with a primitive form of conflict learning. Before releasing **Geo**, we had experimented with naive matching, the algorithm in [10], and many ad hoc methods. Matching with conflict learning is the only approach that results in acceptable performance. Despite this, matching was still a critical operation in the last two-valued version of **Geo**. In the two-valued version of **Geo**, lemmas had form  $v_1/x_1, \dots, v_n/x_n \rightarrow \perp$ , i.e. they had form  $(\bar{v}/\bar{x}) \rightarrow \perp$  for a single subtllet.

In [5] we proposed to replace the lemmas of **Geo** 2007 by arbitrary sets of subtllets. It is quite easy to see, that in general such a lemma can be in conflict with more substitutions than a lemma of the previous form. For example, if we assume that the domain is  $\{X, Y, Z\}$  and the range  $\{0, 1, 2\}$ , then  $(X, Y, Z)/(0, 1, 2) \rightarrow \perp$  rejects a single substitution, while

$(X, Y, Z)/(0, 1, 2)$ ,  $(X, Y, Z)/(2, 1, 0)$  rejects 25 substitutions. Since in case of a conflict, one can always obtain a lemma of the second form, it seemed that lemmas of the second form should be preferred over lemmas of the first form.

The latest version of **Geo** see ([15]) used the algorithm of [5] with lemmas of the unrestricted form above. Although this matching algorithm performs better than matching in **Geo** 2007f, recent experiments have shown that it performs significantly worse than some other approaches, in particular translation to SAT and Algorithm 3.3 in combination with flat lemmas. Flat lemmas are lemmas of form  $v_1 \in V_1 \vee \dots \vee v_n \in V_n$ . Surprisingly, Algorithm 3.3 with unrestricted lemmas performs several orders worse than Algorithm 3.3 with flat lemmas. This is rather surprising, because every general lemma can be flattened into a lemma of the second form by picking a single assignment from each substlet. The resulting lemma is obviously less general than its original, non-flattened version. This loss of generality also applies to the reasoning rules that we use on lemmas. If two substlets in two general lemmas are in conflict, then their flattenings are not necessarily in conflict. Conversely, whenever two flattened substlets are in conflict, their original counterparts are. This means that by using flattened lemmas, one loses conflicts with substitutions, and also resolution derivations involving lemma resolution. Despite this clever reasoning, the first columns of Figure 8 show that Algorithm 3.3 with flat lemmas performs approximately 200-400 times worse than Algorithm 3.3 with unrestricted lemmas. One could assume that this is caused by the fact that handling of unrestricted lemmas is more costly, and that their theoretical advantage is compensated by the increased cost of their maintenance. This assumption is rejected by Figure 8, because Algorithm 3.3 with flattened lemmas is not only faster, but it also uses less lemmas, typically by a factor 2-3. The only point where Algorithm 3.3 with and without flattening can diverge, is when a conflict lemma rejects a substitution  $\Theta$ , and there exists more than one conflict lemma. Since both versions will prefer the shortest lemma. Hence it must occur that flattening changes the relative sizes of lemmas.

The outcomes of the experiments make it probable that the best approach to matching will be either Algorithm 3.3 with flat lemmas, or translation to SAT in combination with a SAT-solver, which we will describe in Section 6.

From the practical point of view, the fact that the refining algorithm in [5] turned out not competitive, is not a serious loss. Despite being elegant on paper, it was hard to implement. Implementation of Algorithm 3.3 was much easier, and in the long term, it is better that the easier algorithm has the better performance. Moreover, it is clear from Figure 8 that matching in future versions of **Geo** can be approximately 1000 times faster than it was at **Geo** 2016c ([15]).

We will now introduce the flattened lemmas, and prove that Algorithm 3.3 can always generate such a conflict lemma.

**Definition 4.1.** A *lemma* is an object of form  $\{v_1/V_1, \dots, v_n/V_n\}$  with  $n \geq 0$ . The  $v_i$  are variables, and the  $V_i$  are finite sets of constants.

It is convenient to treat lemmas as total functions from variables to sets of constants. For a variable  $v$  and  $\lambda = \{v_1/V_1, \dots, v_n/V_n\}$ ,  $\lambda(v)$  is defined as  $\bigcup \{V_i \mid v_i = v\}$ .

Let  $\Theta$  be a substitution. We say that  $\Theta$  *makes  $\lambda$  true* if there exists a variable  $v$  in the domain of  $\Theta$ , for which  $v\Theta \in \lambda(v)$ .

We say that  $\Theta$  *makes  $\lambda$  false* if all variables  $v$  for which  $\lambda(v)$  is nonempty, are in the domain of  $\Theta$ , and  $v\Theta \notin \lambda(v)$ . In that case, we write  $\Theta \models \neg\lambda$ .

**Definition 4.2.** Let  $(\Sigma^+, \Sigma^-)$  be a GCSP. Let  $\lambda$  be a lemma. We say that  $\lambda$  is *valid in*  $(\Sigma^+, \Sigma^-)$  if every solution  $\Theta$  of  $(\Sigma^+, \Sigma^-)$  makes  $\lambda$  true.

For a given substitution  $\Theta$ , we call  $\lambda$  a *conflict lemma* if  $\lambda$  is valid and  $\Theta$  makes  $\lambda$  false.

If  $\Theta$  is a substitution, and there exists a valid lemma that is false in  $\Theta$ , then it is not possible to extend  $\Theta$  into a solution of  $(\Sigma^+, \Sigma^-)$ .

In order to derive the conflict lemma, the following rules will be used:

**Definition 4.3.** Given a GCSP  $(\Sigma^+, \Sigma^-)$ , we define the following derivation rules:

**RESOLUTION:** Let  $\lambda_1, \dots, \lambda_m$  be a sequence of lemmas. Let  $v$  be a variable. Let  $V$  be the set of variables  $v$ , for which one of the  $\lambda_j$  has  $\lambda(v) \neq \emptyset$ . We define the *v-resolvent of  $\lambda_1, \dots, \lambda_m$*  as

$$\{v / \bigcap_{1 \leq j \leq m} \lambda_j(v)\} \cup \{v' / \bigcup_{1 \leq j \leq m} \lambda_j(v') \mid v' \in V \wedge v' \neq v\}.$$

**PROJECTION:** Let  $c \in \Sigma^+$  be a clause, let  $\lambda$  be a lemma. We call  $\lambda$  a *projection* of  $c$ , if every substlet  $(\bar{v}/\bar{x}) \in c$  contains an assignment  $v/x$ , s.t.  $x \in \lambda(v)$ .

**$\sigma$ -RESOLUTION:** Let  $\sigma \in \Sigma^-$  be a blocking. Write  $\sigma = \{v_1/x_1, \dots, v_n/x_n\}$  ( $n > 0$ ). Let  $c_1, \dots, c_n \in \Sigma^+$  be clauses, chosen in such a way that every variable  $v_i$  occurs in  $c_i$ . For every  $c_i$ , let

$$V_i = \{x \mid c_i \text{ contains a substlet } \bar{w}/\bar{y} \text{ which contains } v_i/x \text{ and } x \neq x_i\}.$$

Then the lemma

$$\{v_1/V_1, \dots, v_n/V_n\}$$

is called a  *$\sigma$ -resolvent of  $c_1, \dots, c_n$* .

The lemmas  $\{x/\{1, 2, 3\}, y/\{2, 3\}\}$  and  $\{x/\{3, 4\}, y/\{3, 4\}, z/\{2\}\}$  can resolve into  $\{x/\{3\}, y/\{2, 3, 4\}, z/\{2\}\}$ .

Given clauses  $c_1 = \{(x, y)/(1, 2), (x, y)/(1, 1), (x, y)/(3, 3)\}$  and  $c_2 = \{(y, z)/(1, 2), (y, z)/(2, 1)\}$ , and a blocking  $(x, z)/(1, 2)$ , one can obtain the  $\sigma$ -resolvent  $\{x/\{3\}, z/\{1\}\}$ .

The lemma  $\lambda'_1 = \{x/\{1, 3\}\}$  is a projection of  $c_1$ .  $\lambda''_1 = \{x/\{3\}, y/\{1, 2\}\}$  is also a projection of  $c_1$ .

It is easy to see that the reasoning rules are valid, which implies that every lemma that has been obtained by repeated application from the original clauses in  $\Sigma^+$  and blockings in  $\Sigma^-$ , is valid.

**Lemma 4.4.** Let  $(\Sigma^+, \Sigma^-)$  be a GCSP. Let  $\Theta$  be an interpretation. Let  $\sigma \in \Sigma^-$  be a blocking for which  $\Theta \models \sigma$ . Let  $\lambda$  be a  $\sigma$ -resolvent of  $\sigma$ . Then  $\Theta$  makes  $\lambda$  false.

*Proof.* Write  $\sigma = \{v_1/x_1, \dots, v_n/x_n\}$ . Let  $c_1, \dots, c_n \in \Sigma^+$  be the clauses that were used in the construction of  $\lambda$ . Because  $\Theta \models \sigma$ , we know that for every  $i$  ( $1 \leq i \leq n$ ), we have  $v_i\Theta = x_i$ .

From the construction of the  $V_i$ , it follows that  $x_i \notin V_i$ . Because the variables  $v_i$  are pairwise distinct, we have  $V_i = \lambda(x_i)$ . It follows that  $x_i \notin \lambda(v_i)$ .

For all other variables  $v$  that do not occur in  $\sigma$ , we have  $\lambda(v) = \emptyset$ . We can conclude that if  $\lambda(v)$  is non-empty, then  $v$  equals one of the  $v_i$ , and we have  $v_i\Theta \notin \lambda(v_i)$ .  $\square$

The following lemma states that substlets that are switched off, were switched off because they conflict  $\Theta$ , possibly with help of a blocking.

**Lemma 4.5.** *At every moment during Algorithm 3.3, for every refinement  $(c_i \Rightarrow d_i) \in \overline{C}$ , the following holds: If  $s \in (d_i \setminus c_i)$ , then either  $s$  is in conflict with  $\Theta$ , or  $\Theta \cup \{s\} \models \sigma$ , for a  $\sigma \in \Sigma^-$ .*

*Proof.* There are two points at which refinement can take place, **PICK 1** and **FORW 1a**. At **PICK 1**, clause  $c_i$  is refined into  $\overline{v}_j/\overline{x}_j$ , after which  $\Theta$  is extended with  $\overline{v}_j/\overline{x}_j$ . If some substlet  $s$  occurs in  $c_i \setminus \{\overline{v}_j/\overline{x}_j\}$ , then either  $s \in c_i \setminus d_i$ , or  $s \in d_i \setminus \{\overline{v}_j/\overline{x}_j\}$ . In the first case, the desired property is inherited from the previous state, because it is an invariant. In the second case, because  $\Theta$  is extended by  $\overline{v}_j/\overline{x}_j$  at the same time, we can apply Lemma 2.6.

At **FORW 1a**, if  $s \in d'' \setminus c_i$ , then either  $s \in d_i \setminus c_i$ ,  $s \in d' \setminus d_i$ , or  $s \in d'' \setminus d'$ . In the first case, the desired property is inherited from the previous state. In the second case, it follows from the construction of  $d'$ , that  $s$  was in conflict with  $\Theta$ . In the third case, it follows from the construction of  $d''$ , that there is a  $\sigma \in \Sigma^-$ , for which  $\Theta \cup \{s\} \models \sigma$ .  $\square$

The following property is the essential property, for proving that Algorithm 3.3 can always return a conflict lemma.

**Lemma 4.6.** *Let  $(\Sigma^+, \Sigma^-)$  be a GCSP. Let  $c \in \Sigma^+$  be a clause. Let  $\Theta$  be a substitution. Let  $\Lambda$  be a set of lemmas. Assume that there is no  $\sigma \in \Sigma^-$ , s.t.  $\Theta \models \sigma$ , and no  $\lambda \in \Lambda$ , s.t.  $\Theta$  makes  $\lambda$  false. Assume that for every substlet  $s \in c$ , either*

- (1)  $s$  is in conflict with  $\Theta$ ,
- (2)  $\Theta \cup \{s\} \models \sigma$ , for a  $\sigma \in \Sigma^-$ , or
- (3)  $\Theta \cup \{s\}$  makes a  $\lambda \in \Lambda$  false.

*Then it is possible to derive a conflict lemma for  $\Theta$  from  $\Sigma^+$  and  $\Lambda$ , by applying the rules in Definition 4.3.*

*Proof.* We first remove (2) by means of  $\sigma$ -resolution. We will add the resulting  $\sigma$ -resolvents to  $\Lambda$ .

For every  $s \in c$ , for which (1), (3) do not apply, (2) must apply. Write  $\sigma = \{v_1/x_1, \dots, v_n/x_n\}$ . Since  $(\Sigma^+, \Sigma^-)$  is range restricted, we can find clauses  $c_1, \dots, c_n \in \Sigma^+$ , s.t. each  $v_i$  occurs in  $c_i$ . We now can construct the  $\sigma$ -resolvent. Write  $\lambda$  for the resulting lemma. It follows from Lemma 4.4 that  $\lambda$  is false in  $\Theta$ . We can add  $\lambda$  to  $\Lambda$ . At this point, we have for every  $s \in c$ , either (1) or (3). The rest of the proof is Lemma 4.7.  $\square$

**Lemma 4.7.** *Let  $(\Sigma^+, \Sigma^-)$  be a GCSP. Let  $c \in \Sigma^+$  be a clause. Let  $\Theta$  be a substitution. Let  $\Lambda$  be a set of lemmas. Assume that there is no  $\lambda \in \Lambda$ , s.t.  $\Theta$  makes  $\lambda$  false. Assume that for every substlet  $s \in c$ , either*

- (1)  $s$  is in conflict with  $\Theta$ , or
- (2)  $\Theta \cup \{s\}$  makes a  $\lambda \in \Lambda$  false.

*Then it is possible, using the rules in Definition 4.3, to obtain a conflict lemma for  $\Theta$  from  $c$  and  $\Lambda$ .*

*Proof.* We prove the lemma by induction on the number of unassigned variables in  $c$ . Let  $c_1$  be the part of  $c$  to which (1) applies, and let  $c_2 = c \setminus c_1$ .

Since each  $s \in c_1$  is in conflict with  $\Theta$ , one can obtain a projection  $\mu_1$  of  $c_1$  by picking from each  $s \in c_1$  an assignment  $v/x$  for which  $v\Theta$  is defined and  $v\Theta \neq x$ . By construction,  $\mu_1$  will be false in  $\Theta$ .

If there are no unassigned variables in  $c$ , then  $c_2$  must be empty. This means that  $\mu_1$  is a projection of  $c$  and false in  $\Theta$ , so we are done.

Otherwise, select a  $v$  in  $c$  that is unassigned by  $\Theta$ . Let  $V$  be set of values that are assigned to  $v$  by the substlets in  $c_2$ . Define  $\mu_2 = \{v/V\}$ . Clearly,  $\mu_2$  is a projection of  $c_2$ , and  $\mu = \mu_1 \cup \mu_2$  is a projection of  $c$ . For each value  $x \in V$ , define  $\Theta_x = \Theta \cup \{v/x\}$ . If there is no  $\lambda \in \Lambda$ , that is false in  $\Theta_x$ , then  $c$ ,  $\Theta_x$ ,  $\Lambda$  still satisfy the conditions of Lemma 4.7. Moreover, since  $\Theta_x$  contains an assignment to  $v$ , the number of unassigned variables in  $c$  has decreased by one. This means that we can assume, by induction, that we can derive a lemma  $\lambda_x$  that is false in  $\Theta_x$ . If  $\lambda_x$  is also a conflict lemma of  $\Theta$ , we have completed the proof. Otherwise, we can assume that  $\lambda_x$  is added to  $\Lambda$ .

At this point,  $\Lambda$  contains a conflict lemma  $\lambda_x$  for every  $\Theta \cup \{v/x\}$  with  $x \in V$ . Let  $\lambda$  be the  $v$ -resolvent of the projection  $\mu$  constructed above, and the  $\lambda_x$ , i.e.

$$\lambda = \{ v / ( \mu(v) \cap \bigcap_{x \in V} \lambda_x(v) ) \} \cup \{ v' / ( \mu(v') \cup \bigcup_{x \in V} \lambda_x(v') ) \mid v' \neq v \}.$$

In order to show that  $\lambda$  is false in  $\Theta$ , we have to show that for every variable  $v'$ , for which  $\lambda(v') \neq \emptyset$ ,  $v'\Theta$  is defined, and  $v'\Theta \notin \lambda(v')$ .

- For  $v$ , we just show that  $\lambda(v) = \emptyset$ . We have  $\mu(v) = \mu_2(v)$ , because  $\mu_1(v) = \emptyset$ . It follows from the fact that  $v$  is undefined in  $\Theta$ , and  $\mu_1$  is false in  $\Theta$ . For each  $x \in \mu_2(v)$ , we know that  $\lambda_x$  is false in  $\Theta \cup \{v/x\}$ , which implies that  $x \notin \lambda_x(v)$ . This implies that  $x$  is not in the intersection of all  $\lambda_x(v)$ , which in turn implies  $\mu(v)$  and  $\bigcap_{x \in V} \lambda_x(v)$  have no elements in common.
- If  $v' \neq v$  and  $\lambda(v') \neq \emptyset$ , then either  $\lambda_x(v') \neq \emptyset$ , for an  $x \in V$ , or  $\mu(v') \neq \emptyset$ . In the first case, it follows from the fact that  $\lambda_x$  is false in  $\Theta \cup \{v/x\}$  and  $v' \neq v$ , that  $v'\Theta$  is defined. In the second case, we know that  $\mu_2(v')$  only assigns to  $v$ , so that  $\mu_1(v') \neq \emptyset$ . Since we know that  $\mu_1$  is false in  $\Theta$ , we know  $v'\Theta$  is defined.

At this point, we are certain that  $v'\Theta$  is defined, so that we can start showing that  $v'\Theta \notin \mu(v') \cup \bigcup_{x \in V} \lambda_x(v')$ . If  $v'\Theta \in \mu(v')$ , then, because  $\mu_2$  only assigns to  $v$ , we have  $v'\Theta \in \mu_1(v')$ . This is impossible because  $\mu_1$  is false in  $\Theta$ .

We can also not have  $v'\Theta \in \lambda_x(v')$ , for any  $x \in V$ , because this would imply that  $v'(\Theta \cup \{v/x\}) \in \lambda_x(v')$ , which contradicts the fact that  $\lambda_x$  is false in  $\Theta \cup \{v/x\}$ .

□

At this point, it is straightforward to prove that Algorithm 3.3 can always derive a conflict lemma. There are two points in Algorithm 3.3 where the substitution is extended. We show for both points that it is possible to obtain a conflict lemma when the substitution is restored.

**FORW 1b:** The substitution  $\Theta$  is extended by the common assignments in  $d''$ . Since the extension of  $\Theta$  had a conflict lemma, we know that for each  $s \in d''$ ,  $\Theta \cup \{s\}$  has a conflict lemma. It follows from Lemma 4.5 that for every substlet  $s$  in  $d'' \setminus \{c_i\}$ , either  $s$  is in conflict with  $\Theta$ , or  $\Theta \cup \{s\}$  implies  $\sigma$ , for a blocking  $\sigma \in \Sigma^-$ . From Lemma 3.4, we know that there is no  $\sigma \in \Sigma^-$ , s.t.  $\Theta \models \sigma$ . It follows that we can apply Lemma 4.6 with  $\Lambda = \{\lambda\}$  to obtain a conflict lemma for  $\Theta$ .

**PICK:** Let  $c_i \Rightarrow d_i$  be the refinement that was selected by PICK. Let  $\Lambda$  be the set of conflict lemmas that were returned by the recursive calls of **findmatch**. If there is a  $\lambda \in \Lambda$  that is false in  $\Theta$ , we can return  $\lambda$ . Otherwise, we know that no  $\lambda \in \Lambda$  is false in  $\Theta$ . From Lemma 3.4, we know that there is no  $\sigma \in \Sigma^-$ , s.t.  $\Theta \models \sigma$ .

By Lemma 4.5, every substlet  $s \in (c_i \setminus d_i)$ , is either in conflict with  $\Theta$ , or there exists a  $\sigma \in \Sigma^-$ , s.t.  $\Theta \cup \{s\} \models \sigma$ . This implies that we can apply Lemma 4.6 to obtain a conflict lemma of  $\Theta$ .

In an implementation of Algorithm 3.3, there is no need to follow the rules of Definition 4.3 carefully, because the conflict lemma can be constructed immediately from the premisses of Lemma 4.6.

In order make Algorithm 3.3 reuse conflict lemmas, one has to add before **FORW 1**: If there is a  $\lambda \in \Lambda$  containing variable  $v$ , s.t.  $\Theta$  makes  $\lambda$  false, then return  $\lambda$ .

Integrating lemmas into the refining step of **FORW 1** seems difficult, because the notion of connection (Definition 3.2) must be extended to include ' $v$  and  $w$  occur together in a lemma  $\lambda \in \Lambda$ .' ) Currently we don't know how to efficiently enumerate variables that are connected through a lemma. Algorithm 7.4 will not have this problem, because it can use two variable watching.

## 5. MATCHING BASED ON LOCAL CONSISTENCY CHECKING

We will discuss the matching algorithm of [5]. Its performance turned out not competitive, so we will omit most of the details, in particular the completeness proofs for learning. The algorithm is based on the fact that local consistency checking rejects a large percentage of GCSPs without backtracking.

Local consistency checking is the following procedure: For every clause  $c = \{s_1, \dots, s_n\} \in \Sigma^+$ , check, for all sets of clauses  $C$  with size  $|S| \geq 1$ , if  $\{s_i\} \cup C$  has a solution. If not, then remove  $s_i$  from  $c$ . Keep on doing this, until no further changes are possible or a clause has become empty. The procedure is described in detail in Section 10. Local consistency checking with small  $S$  rejects a large percentage of instances without backtracking. It therefore seemed reasonable to combine local consistency checking with backtracking in the following way:

**FILTER:** Apply local consistency checking. If this results in an empty clause, then backtrack to the last decision. If there are no decisions left, then report failure.

**DECIDE:** If every clause has become unit, then report a solution. Otherwise, pick a non-unit clause, and replace it by a singleton consisting of one of its substlets. Continue at **FILTER**. If this results in an empty clause, then backtrack through the remaining substlets of the clause.

The assumption was that local consistency checking could play the same role as unit propagation in DPLL, and that local consistency checking would be equally effective on the subproblems obtained during backtracking, as on the initial problem. This assumption turned out false. In [5], the algorithm is described for  $S = 1$ , but we have implemented it for arbitrary  $S \geq 1$ . Note that a size of  $S$  means that  $\|C\| = S$ , so that  $\{c_i\} \cup C$  has size  $S + 1$ . Performance results are presented in Figure 8 in Section 8. It can be seen that,  $S > 1$  does not perform better than  $S = 1$ . It rarely creates less lemmas, and it usually costs more time.

The main observation to made is that the algorithm is not close to being competitive against Algorithm 3.3 with flat lemmas, or translation to SAT. In addition to that, it turned out rather unpleasant to implement, much harder than Algorithm 3.3. Especially  $S > 1$  is difficult to handle, because the resolution rules for obtaining lemmas become rather complicated. This does not only apply to the implementation, but also to the theoretical description.

We define the lemmas that were used by the matching algorithm, and the reasoning rules that it uses. A clause can be viewed as a special form of lemma in which the substlets have the same domain.

**Definition 5.1.** A *lemma* is a finite set of substlets, possibly with different domains.

If  $\lambda$  is a lemma, and  $\Theta$  a substitution, then  $\Theta$  makes  $\lambda$  true if there is a substlet  $(\bar{v}/\bar{x}) \in \lambda$ , s.t.  $\Theta$  makes  $\lambda$  true.  $\Theta$  makes  $\lambda$  false if every substlet  $(\bar{v}/\bar{x}) \in \lambda$  is in conflict with  $\Theta$ . We say that  $\lambda$  is *valid* relative to  $(\Sigma^+, \Sigma^-)$ , if  $\Theta$  is true in every solution  $\Theta$  of  $(\Sigma^+, \Sigma^-)$ . We call  $\lambda$  a *conflict lemma* if  $\lambda$  is false in the current  $\Theta$  and valid  $(\Sigma^+, \Sigma^-)$ .

Learning was based on the following resolution rules:

**Definition 5.2.** Let  $\lambda_1$  and  $\lambda_2$  be lemmas. Let  $\mu_1 \subseteq \lambda_1$ , and let  $\mu_2 \subseteq \lambda_2$ . Assume that every  $s_1 \in \mu_1$  is in conflict with every  $s_2 \in \mu_2$ . Then  $(\lambda_1 \setminus \mu_1) \cup (\lambda_2 \setminus \mu_2)$  is a resolvent of  $\lambda_1$  and  $\lambda_2$ .

One can resolve  $\lambda_1 = \{ (x, y)/(1, 2), (x, y)/(1, 1), (x, y)/(3, 3) \}$  with  $\lambda_2 = \{ (y, z)/(1, 2), (y, z)/(2, 1) \}$  based on  $\mu_1 = \{ (x, y)/(1, 2), (x, y)/(3, 3) \}$ , and  $\mu_2 = \{ (y, z)/(1, 2) \}$ . The resolvent is  $\{ (x, y)/(1, 1), (y, z)/(2, 1) \}$ .

**Definition 5.3.** Let  $\sigma \in \Sigma^-$  be a blocking. Let  $c_1, \dots, c_n \in \Sigma^+$  be a sequence of clauses containing all variables of  $\sigma$ . For each  $c_i$ , let  $\rho_i = \{ s \in c_i \mid s \text{ is in conflict with } \sigma \}$ . Then  $\rho_1 \cup \dots \cup \rho_n$  is a  $\sigma$ -*resolvent* of  $c_1, \dots, c_n$ .

Using  $\lambda_1, \lambda_2$  given above, and blocking  $(x, z)/(1, 2)$ , one can obtain the  $\sigma$ -resolvent  $\{ (x, y)/(3, 3), (y, z)/(2, 1) \}$ .

It is easy to see that both conflict resolution and  $\sigma$ -resolution are valid reasoning rules, which implies that every lemma that was derived by repeated application of resolution from the original clauses in  $\Sigma^+$ , is valid.

In [5], it was shown that a matching algorithm using  $S = 1$  can always obtain a conflict lemma using resolution and  $\sigma$ -resolution. For  $S > 1$ , an additional rule, called *product resolution* is required. Results are listed in Figure 8.

After observing that Algorithm 3.3 improves by a factor 500 when lemmas are flattened, we tried the same with the refining algorithm. Whenever a new lemma is derived, the assignments that do not contribute to conflicts are removed from the substlets. Different from Algorithm 3.3, this does not necessarily lead to a lemma consisting only of single-assignment substlets, but in most cases it does. Surprisingly, this has a strong, negative impact on the performance.

## 6. TRANSLATION TO SAT

Translating an instance of the matching problem to SAT is easy, and modern SAT solvers have become very effective. As a consequence, translation to SAT should be attempted. In this section, we give two methods of translating

GCSP into SAT. The translations are not complicated, and MiniSat [9] performs rather well on the results of the translations. Results are listed in the last two columns of Figure 8 in Section 8. It is quite possible that, even after reimplementing of Algorithm 3.3 with improved datastructures, and implementation of Algorithm 7.4, it will turn out that some combination of filtering (see Section 10) and translation to SAT is the best approach.

In the first translation only substlets are translated. We assign propositional variables to the substlets, specify that at least one substlet from each clause has to be selected, and list the conflicts between the substlets.

**Definition 6.1.** We assume a general mapping  $[ \ ]$  that transforms mathematical objects into distinct propositional variables.

**Definition 6.2.** Let  $(\Sigma^+, \Sigma^-)$  be GCSP. The translation into propositional logic has form  $(A, P)$ , where  $A$  is a set of atoms, and  $P$  is a set of clauses over  $A$ . Assume that the GCSP has form  $(\Sigma^+, \Sigma^-)$ , assume that  $\Sigma^+$  contains  $n$  clauses, and write  $\{s_{i,1}, \dots, s_{i,k_i}\}$  for the  $i$ -th clause of  $\Sigma^+$ .

The set of atoms is defined as  $A = \{[s_{i,j}] \mid 1 \leq i \leq n, 1 \leq j \leq k_i\}$ . The clause set  $P$  is defined as follows:

- (1) For every  $c_i = \{s_{i,1}, \dots, s_{i,k_i}\} \in \Sigma^+$  ( $1 \leq i \leq n$ ), the propositional clause set  $P$  contains the propositional clause  $\{[s_{i,1}], \dots, [s_{i,k_i}]\}$ , and for every  $j_1, j_2$  ( $1 \leq j_1 < j_2 \leq k_i$ ) the clause  $\{\neg[s_{i,j_1}], \neg[s_{i,j_2}]\}$ .
- (2) For every pair of distinct clauses  $c_{i_1}, c_{i_2} \in \Sigma^+$  that share a variable, for every substlet  $s \in c_{i_1}$ ,  $P$  contains the clause  $\{\neg[s] \} \cup \{[s'] \in c_{i_2} \mid s' \in c_{i_2}, \text{ and } s' \text{ is not in conflict with } s \}$ .
- (3) For every blocking  $\sigma \in \Sigma^-$ , we assume that there is a way of selecting a most suitable subset  $C_\sigma$  of  $\Sigma^+$  that contains all variables of  $\sigma$ . Then  $P$  contains the clause  $\{[s] \mid \exists c \in C_\sigma, \text{ s.t. } s \in c \text{ and } s \text{ is in conflict with } \sigma \}$ .

The first part specifies that exactly one substlet must be selected from each  $c \in \Sigma^+$ . The second part specifies that if one selects a substlet  $s$  from  $c_{i_1}$ , one has to select a substlet  $s'$  from  $c_{i_2}$  that is not in conflict with  $s$ . The third part of Definition 6.2 can be viewed as an application of  $\sigma$ -RESOLUTION (Definition 5.3).

The second translation differs from the first translation in the fact that it does not only translate substlets, but also variable assignments. In addition to the substlets, it assigns propositional variables to variable assignments  $v/x$ . It specifies the dependencies between substlets and variable assignments. Instead of relying on  $\sigma$ -RESOLUTION, blockings can be specified directly in terms of the forbidden variable assignments.

**Definition 6.3.** Let  $(\Sigma^+, \Sigma^-)$  be a GCSP. Write  $\Sigma^+ = \{c_1, \dots, c_n\}$ . Write each  $c_i$  in the form  $\{s_{i,1}, \dots, s_{i,k_i}\}$ . The translation to propositional logic has form  $(A, P)$ , where  $A$  is the set of atoms used in the translation, and  $P$  is the set of clauses. The set of atoms  $A$  is defined as

$$\{[s_{i,j}] \mid 1 \leq i \leq n, 1 \leq j \leq k_i\} \cup \{[v/x] \mid (v/x) \text{ occurs in a substlet in } \Sigma^+\}.$$

The set of propositional clauses  $P$  is obtained as follows:

- (1) For every clause  $c_i$ , ( $1 \leq i \leq n$ ), clause set  $P$  contains the clause  $\{[s_{i,1}], \dots, [s_{i,k_i}]\}$ .
- (2) For every substlet  $s_{i,j}$  with  $1 \leq i \leq n, 1 \leq j \leq k_i$ , for every assignment  $v/x$  that occurs in  $s_{i,j}$ , clause set  $P$  contains the clause  $\{\neg[s_{i,j}], [v/x]\}$ .
- (3) For every variable  $v$  that occurs in  $\Sigma^+$ , for every two distinct values  $x_1, x_2$ , s.t.  $v/x_1$  and  $v/x_2$  occur somewhere in substlets in  $\Sigma^+$ , clause set  $P$  contains the clause  $\{\neg[v/x_1], \neg[v/x_2]\}$ .
- (4) For every blocking  $\sigma \in \Sigma^-$ , if every  $(v/x) \in \sigma$  occurs somewhere in a clause in  $\Sigma^+$ , then clause set  $P$  contains the clause  $\{\neg[v/x] \mid (v/x) \in \sigma\}$ . If some  $(v/x) \in \sigma$  does not occur in  $\Sigma^+$ , then  $\sigma$  is impossible, and there is no need to generate a clause for it.



We show correctness of Definition 6.3. If  $(\Sigma^+, \Sigma^-)$  has a solution  $\Theta$ , one can define a satisfying interpretation  $I$  for  $(A, P)$  as follows:

- For  $1 \leq i \leq n$ ,  $1 \leq j \leq k_i$ , set  $I([s_{i,j}]) = \mathbf{t}$  iff  $\Theta \models s_{i,j}$ .
- For every assignment  $v/x$  occurring in a sublet  $s$  occurring in a clause  $c_i$ , set  $I([v/x]) = \mathbf{t}$  iff  $v\Theta = x$ .

It is easily checked that  $I$  makes all clauses in Definition 6.3 true.

For the other direction, assume that  $(A, P)$  has a satisfying interpretation  $I$ . Define  $\Theta = \{ (v/x) \mid I([v/x]) = \mathbf{t} \}$ . By part 4,  $\Theta$  does not contain conflicting assignments. By part 1 and part 2,  $\Theta$  contains an assignment for every variable occurring in  $\Sigma^+$ . Because of part 3,  $\Theta$  does not imply a blocking  $\sigma \in \Sigma^-$ . By part 1 and part 2, every  $c_i \in \Sigma^+$  contains one sublet that agrees with  $\Theta$ .

We will end with an example of both translations:

**Example 6.4.** We will translate the following GCSP. As usual,  $\Sigma^+$  and  $\Sigma^-$  are separated by a horizontal bar.

$$\begin{array}{c} (X, Y) / (0, 1) \mid (1, 0) \\ (Y, Z) / (0, 0) \mid (0, 1) \mid (1, 0) \\ \hline (X, Z) / (0, 0) \\ (X, Z) / (1, 1) \end{array}$$

$\Sigma^+$  alone has three solutions:

$$\begin{aligned} \Theta_1 &= \{ X := 0, Y := 1, Z := 0 \}, \\ \Theta_2 &= \{ X := 1, Y := 0, Z := 0 \}, \\ \Theta_3 &= \{ X := 1, Y := 0, Z := 1 \}. \end{aligned}$$

The first solution is blocked by  $(X, Z)/(0, 0)$ , the third solution is blocked by  $(X, Z)/(1, 1)$ , so that only  $\Theta_2$  is a solution of the complete GCSP. Assume that

$$[(X, Y)/(0, 1)] = 1, [(X, Y)/(1, 0)] = 2, [(Y, Z)/(0, 0)] = 3, [(Y, Z)/(0, 1)] = 4, [(Y, Z)/(1, 0)] = 5.$$

Definition 6.2 constructs the following translation:

$$\text{Part 1 : } \begin{pmatrix} 1 & 2 \\ 3 & 4 & 5 \\ -1 & -2 \\ -3 & -4 \\ -3 & -5 \\ -4 & -5 \end{pmatrix} \quad \text{Part 2 : } \begin{pmatrix} -1 & 5 \\ -2 & 3 & 4 \\ -3 & 2 \\ -4 & 2 \\ -5 & 1 \end{pmatrix} \quad \text{Part 3 : } \begin{pmatrix} 1 & 4 \\ 2 & 3 & 5 \end{pmatrix}$$

The only satisfying interpretation is  $\{-1, 2, 3, -4, -5\}$ , which corresponds to  $\Theta_2$ . In order to apply the second translation, assume that

$$[X/0] = 6, [X/1] = 7, [Y/0] = 8, [Y/1] = 9, [Z/0] = 10, [Z/1] = 11.$$

The second translation constructs

$$\text{Part 1 : } \begin{pmatrix} 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \quad \text{Part 2 : } \begin{pmatrix} -1 & 6 \\ -1 & 9 \\ -2 & 7 \\ -2 & 8 \\ -3 & 8 \\ -3 & 10 \\ -4 & 8 \\ -4 & 11 \\ -5 & 9 \\ -5 & 10 \end{pmatrix} \quad \text{Part 3 : } \begin{pmatrix} -6 & -7 \\ -8 & -9 \\ -10 & -11 \end{pmatrix} \quad \text{Part 4 : } \begin{pmatrix} -6 & -10 \\ -7 & -11 \end{pmatrix}$$

Its only satisfying interpretation is  $\{-1, 2, 3, -4, -5, -6, 7, 8, -9, 10, -11\}$ , which again corresponds to  $\Theta_2$ .

## 7. MATCHING BY SUBSTITUTION REFINEMENT

In this section, we present another matching algorithm that is based on a form of refinement. Instead of extending the substitution, as Algorithm 3.3 does, the new algorithm refines the substitution simultaneously with the clauses. It has found a solution when the substitution and the clauses are unary. Initially, the substitution assigns to each variable the set of all possible values. In order to do this, we define a data structure called *substitution stack*. A substitution stack is similar to a refinement stack.

**Definition 7.1.** A *substitution stack*  $\Theta$  is a finite sequence of assignments of form  $v_i/V_i$ , where each  $v_i$  is a variable, and each  $V_i$  is a finite set of constants.  $\Theta$  must satisfy the following requirements:

- For each variable  $v$ , there is at least one assignment  $v_i/V_i$  with  $v_i = v$  in  $\Theta$ .
- If  $i < j$  and  $v_i = v_j$ , then  $V_j \subset V_i$ .

For a variable  $v$ , if  $v_i/V_i$  is the last assignment in  $\Theta$  with  $v = v_i$ , then  $V_i$  is the *current value of*  $v$ . We write  $v\Theta$  for the current value of  $v$ .

We define a predicate  $\alpha_i(\Theta)$  that is true if  $v_i/V_i$  is the last assignment to variable  $v_i$  in  $\Theta$ .

Similar to refinement stacks, an assignment  $v_i/V_i$  can be refined by appending  $v_i/V'$  to  $\Theta$ , for some  $V' \subset V_i$ .

A substitution stack is *unary* if for every  $v$ , the value of  $v\Theta$  is a singleton.

Substitutions stacks can be efficiently implemented in the same way as refinement stacks.

**Definition 7.2.** Let  $\lambda$  be a lemma. Let  $\Theta$  be a substitution stack. We say that  $\Theta$  *makes*  $\lambda$  *true* if there exists a variable  $v$  in the domain of  $\lambda$ , s.t. the last occurrence  $v/V$  of  $v$  in  $\Theta$  has  $V \subseteq \lambda(v)$ .

We say that  $\Theta$  *makes*  $\lambda$  *false*, if for every variable  $v$  in the domain of  $\lambda$ , the last occurrence  $v/V$  of  $v$  in  $\Theta$  has  $\lambda(v) \cap V = \emptyset$ .

We say that  $\Theta$  *makes*  $\lambda$  *productive*, if  $\Theta$  does not make  $\lambda$  true or false, and  $\lambda$  can be written in the form  $\lambda = \mu \cup \{w/W\}$ , s.t.  $\Theta$  makes  $\mu$  false.

Given a GCSP  $(\Sigma^+, \Sigma^-)$  and a substitution stack  $\Theta$ , we call  $\lambda$  a *conflict lemma* if  $\lambda$  is valid in  $(\Sigma^+, \Sigma^-)$ , and false in  $\Theta$ .

The notion of truth and falsehood are the same as in Definition 4.1 for standard substitutions. If a substitution stack  $\Theta$  makes a lemma  $\lambda$  true(or false), then every refinement of  $\Theta$  makes  $\lambda$  true(or false).

The notion of productivity is inspired by *unit propagation* from propositional logic ([12]). Note that, different from propositional logic, when a lemma  $\mu \cup \{w/W\}$  is productive, there still is an assignment  $w/W'$  in  $\Theta$ . In order to be productive, it must be the case that  $W' \not\subseteq W$ , and  $W' \cap W \neq \emptyset$ . In the first case,  $\mu \cup \{w/W\}$  would be true. In the second case,  $\mu \cup \{w/W\}$  would be in conflict. The definition of conflict lemma is analogous to Definition 4.2.

We define the notion of conflict between a substitution stack, and a substlet. The algorithm that we will introduce shortly, will use it in the same way as Algorithm 3.3 to refine clauses.

**Definition 7.3.** Let  $\Theta$  be a substitution stack. Let  $\bar{v}/\bar{x}$  be a substlet. We say that  $\Theta$  and  $\bar{v}/\bar{x}$  are in conflict, if there is a  $(v/x) \in (\bar{v}/\bar{x})$ , s.t.  $x \notin v\Theta$ .

We present Algorithm 7.4 as a recursive algorithm, even when a realistic implementation should use a stack. We present the algorithm as a recursive algorithm, because it is simpler.

**Algorithm 7.4.** A call to **findmatch**( $\bar{C}, k, \Theta, s, \Lambda, \Sigma^-$ ) either returns a conflict lemma, or it refines  $\Theta$  into a solution of  $(\bar{C}, \Sigma^-)$ . Initially, we set  $\bar{C} = (c \Rightarrow c \mid c \in \Sigma^+)$ ,  $k = s = 1$ , and  $\Theta = (v/X \mid v \text{ occurs in } \Sigma^+)$ , where  $X$  is the set of constants occurring in  $\bar{C}$ . The algorithm consists of the following steps:

**FORWARD:** If  $s \leq \|\Theta\|$  or  $k \leq \|\bar{C}\|$ , then call  $\lambda := \mathbf{forward}(\bar{C}, k, \Theta, s, \Lambda, \Sigma^-)$ . If  $\lambda$  is a lemma, then return  $\lambda$ .

**PICK:** Find an  $i$  for which  $\alpha_i(\Theta)$  is true and  $\|V_i\| > 1$ . If no such  $i$  exists, then  $\Theta$  is a solution. Otherwise, partition  $V$  into  $m \geq 2$  non-empty parts  $W_1, \dots, W_j, \dots, W_m$ . It is not required that the  $W_j$  are singletons. For every  $j$  ( $1 \leq j \leq m$ ), do the following:

- (1) Assign  $\Theta := \Theta + (v_i/W_j)$ .
- (2) Recursively call  $\lambda := \mathbf{findmatch}(\bar{C}, k, \Theta, s, \Lambda, \Sigma^-)$ .
- (3) Restore  $\Theta$ . If  $\lambda$  is still false in  $\Theta$ , then return  $\lambda$ .

At this point, each of the recursive calls has returned a conflict lemma. Let  $\lambda_1, \dots, \lambda_m$  be the resulting conflict lemmas. Return  $\text{RES}(v_i, \lambda_1, \dots, \lambda_m)$ .

Algorithm **forward**( $\bar{C}, k, \Theta, s, \lambda, \Sigma^-$ ) tries to refine  $\bar{C}$  and  $\Theta$  as far as possible without backtracking. If it detects a contradiction, it returns a conflict lemma.

**SUBST:** As long as  $s \leq \|\Theta\|$ , let  $v/V$  be the  $s$ -th assignment of  $\Theta$ . If  $\alpha_i(\Theta)$  is false, then proceed immediately to step 5.

- (1) If  $V$  is empty, then return  $\top$ .
- (2) For every  $\lambda \in \Lambda$  that contains  $v$ , do the following: If  $\lambda$  is false in  $\Theta$ , then return  $\lambda$ . If  $\lambda$  is productive, then write  $\lambda = \mu \cup \{w/W\}$ , where  $\mu$  is false in  $\Theta$  and does not contain  $w$ . Append  $w/(W \cap w\Theta)$  to  $\Theta$ .
- (3) If  $V$  is a singleton, then let

$$\Theta' = \{v/x \mid v/\{x\} \in \Theta\}.$$

For every blocking  $\sigma \in \Sigma^-$  that contains variable  $v$ , check if  $\Theta' \models \sigma$ . If yes, then return a  $\sigma$ -resolvent using  $\sigma$ .

(4) For every refinement  $(c_i \Rightarrow d_i)$  in  $\overline{C}$ , s.t.  $d_i$  contains  $v$  and  $\alpha_i(\overline{C})$  is true, let

$$d' = \{s \in d_i \mid s \text{ is not in conflict with } \Theta\}.$$

If  $d' \subset d_i$ , then append  $(c_i \Rightarrow d')$  to  $\overline{C}$ .

(5)  $s := s + 1$ .

**CLAUSES:** As long as  $k \leq \|\overline{C}\|$ , let  $c_k \Rightarrow d_k$  be the  $k$ -th refinement in  $\overline{C}$ .

(1) If  $\alpha_k(\overline{C})$  holds, then for every variable  $v$  occurring in  $d_k$  do the following: Let  $V$  be the set of assignments to  $v$  that occur in the substlets of  $d_k$ . If  $v\Theta \not\subseteq V$ , then append  $v/(v\Theta \cap V)$  to  $\Theta$ .

(2)  $k := k + 1$ .

Before **findmatch** can be called on  $(\Sigma^+, \Sigma^-)$ , one must remove propositional clauses, propositional blockings, and unit blockings as described in Section 2.

It is possible to create the  $\sigma$ -resolvents in advance, for every blocking  $\sigma \in \Sigma^-$ , and insert them into  $\Lambda$ . In that case, **SUBST 3** can be omitted.

We prove, for each of the steps of Algorithm 7.4, that is possible to obtain a conflict lemma:

**SUBST 1:** If  $\Theta$  contains an assignment of form  $v/\emptyset$ , then every lemma is a conflict lemma of  $\Theta$ .

**SUBST 2:** In the first case,  $\lambda$  is obviously a conflict lemma. Otherwise, assume that  $\Theta + w/(W \cap w\Theta)$  has a conflict lemma  $\lambda'$ . If  $\lambda'$  is still false in  $\Theta$ , nothing needs to be done. Otherwise, because the only difference between  $\Theta$  and  $\Theta + w/(W \cap w\Theta)$  is the last assignment to  $w$ , we have  $\lambda'(v') \cap v'\Theta = \emptyset$ , for every variable  $v' \neq w$ . For the productive lemma  $\lambda$ , we also have  $\lambda(v') \cap v'\Theta = \emptyset$ , for every variable  $v' \neq w$ , by the definition of productivity. The  $w$ -resolvent of  $\lambda$  and  $\lambda'$  has form

$$w/(\lambda(w) \cap \lambda'(w)) \cup \{v' / (\lambda(v') \cup \lambda(v)) \mid v' \neq w\}.$$

We show that the resolvent is false in  $\Theta$ . We have shown above that  $v' \neq w$  implies  $v'\Theta \cap (\lambda(v') \cup \lambda(v)) = \emptyset$ . In order to show that  $w\Theta \cap \lambda(w) \cap \lambda'(w) = \emptyset$ , it is sufficient to observe that  $\lambda(w) = W$ .

**SUBST 3:** First use Lemma 4.4 to show that every  $\sigma$ -resolvent  $\lambda$  based on  $\sigma$  is false in  $\Theta'$ .

After that, use the fact that for every variable  $v$  with  $\lambda(v) \neq \emptyset$ , we have  $v\Theta = \{v\Theta'\}$ , to show that  $\lambda$  is false in  $\Theta$  as well.

**CLAUSES 1:** Using projection, we extract a productive lemma  $\lambda$  from  $c_k$ , with which we can proceed as in **SUBST 2**, and obtain the same refinement of  $\Theta$ . If the refined substitution has a conflict lemma  $\lambda'$ , we can resolve it with  $\lambda$  in the same way as in **SUBST 2**.

Since  $c_k$  was previously refined into  $d_k$  based on conflicts with  $\Theta$ , it must be the case that every substlet  $s$  in  $c_k \setminus d_k$  contains an assignment  $v/x$ , for which  $x \notin v\Theta$ . Let  $\lambda_1$  be the projection of  $c_k \setminus d_k$  that is obtained by selecting one such assignment from each substlet in  $c_k \setminus d_k$ . Then  $\lambda = \lambda_1 \cup \{v/V\}$ , with  $V$  being the set of values that are assigned to  $v$  in substlets of  $d_k$ , is a projection of  $c_k$ .

Let  $\mu$  be the subset of  $\lambda_1$  that does not contain  $v$ . Let  $W = \lambda_1(v)$ . We have  $\lambda_1 = \mu \cup \{v/W\}$ . We know that  $v\Theta \cap W = \emptyset$ . Now we can set  $\lambda = \lambda_1 \cup \{v/V\} = \mu \cup \{v/(W \cup V)\}$ . From  $v\Theta \not\subseteq V$  follows that  $v\Theta \not\subseteq (W \cup V)$ , so that  $\lambda$  is either productive or false. If  $\lambda$  is false, we just return  $\lambda$ . Otherwise, since  $v\Theta \cap V = v\Theta \cap (V \cap W)$ , refining with  $\lambda$  has the same effect as refining with  $d_k$ .

**PICK:** Assume that  $\lambda_1, \dots, \lambda_m$  are the conflict lemmas of  $\Theta + (v_i/W_1), \dots, \Theta + (v_i/W_m)$ , that were obtained from the recursive calls. We will show that  $\lambda' = \text{RES}(v_i, \lambda_1, \dots, \lambda_m)$  is false in  $\Theta$ .

Since  $\Theta$  and  $\Theta + (v_i/W_j)$  differ only in the assignment to  $v_i$ , we know that  $v' \neq v_i$  implies that  $v'\Theta \cap \lambda_j(v') = \emptyset$ , for every  $j$  ( $1 \leq j \leq m$ ), so that  $v'\Theta \cap (\lambda_1(v') \cup \dots \cup \lambda_m(v')) = \emptyset$ . We also know that  $\lambda_j(v_i) \cap W_j = \emptyset$ , for every  $j$  ( $1 \leq j \leq m$ ). It follows that  $(\lambda_1(v_i) \cap \dots \cap \lambda_m(v_i)) \cap (W_1 \cup \dots \cup W_m) = \emptyset$ . We have  $v_i\Theta = W_1 \cup \dots \cup W_m$ . As a consequence  $\text{RES}(v_i, \lambda_1, \dots, \lambda_m) =$

$$\{v_i / \bigcap_{1 \leq j \leq m} \lambda_j(v_i)\} \cup \{v' / \bigcup_{1 \leq j \leq m} \lambda_j(v') \mid v' \neq v_i\}$$

is false in  $\Theta$ .

Algorithm 7.4 is currently not implemented, but it can be implemented relatively easily, mostly reusing data structures of Algorithm 3.3. It has in common with the refining algorithm of [5] that it is based on gradual refinement, rather than choice. It avoids some of the pitfalls of the earlier algorithm. It does not use unrestricted lemmas, whose maintenance turned out too costly, it does not involve repeated consistency checks of subsets of clauses, which turned out costly. It allows propagation with productive lemmas, which can be implemented with two-variable watching, and it allows deep backtracking: If during the undoing of a **SUBST 2** step, a conflict lemma is obtained that is productive, one can stop further unwinding, and backtrack to the lowest level on which the lemma is still productive. All of this is no guarantee that Algorithm 7.4 will perform well, but we believe there is some probability.

## 8. EXPERIMENTS

We present measurements on our benchmark set of matching instances. The benchmark sets were obtained by running a naive matching algorithm, and collecting some instances that took more than a day to solve. The horizontal line in the middle of the table marks the fact that the problems in the lower half of the table are from a different problem set, than the problems in the upper half. The problems that are marked with a \* are satisfiable. Entries in Figure 8 have form  $t(\lambda)$ , where  $t$  is the time used in seconds, and  $\lambda$  the number of lemmas generated. For the 3d and 4th column, the times are the CPU-times reported by MiniSat (Version 2.0 beta) ([9]). Since currently MiniSat is not integrated into the program, it is difficult to measure the total time (conversion+solving). For hard problems, the conversion times are probably negligible, but for trivial problems, they may be significant (in the same order of magnitude as the solving times) because translation is quadratic. It can be seen from Figure 8 that translation to propositional SAT is a serious candidate for being the best possible approach, even when we believe that implementation of Algorithm 3.3 can be improved, without using datastructures for the refining algorithms of [5]. We were not sure how to determine the number of lemmas generated during a run of MiniSat, due to the fact that it performs restarts. Currently, we simply added the numbers reported by the different restarts. Since MiniSat probably reuses lemmas between different restarts, this means that the indicated numbers are probably too high. On all problems, except for **mod22**, the translation of Definition 6.2 performs better than Definition 6.3.

Figure 8 shows results for the refining algorithm of [5], discussed in Section 5. It can be seen that using  $S > 1$  is hardly worth the effort, and that the refining algorithm performs

Problem	Algo 3.3	Algo 3.3( <b>flat</b> )	Def 6.2	Def 6.3
<b>mod01</b>	271(42606)	0.93(11688)	0.17(8248)	0.08(3220)
<b>mod02</b>	138(28830)	0.74(8694)	0.19(8248)	0.12(5982)
<b>mod03</b>	80(21822)	0.59(7038)	0.6(20344)	0.17(5288)
<b>mod04</b>	32(14290)	0.27(5145)	0.05(2955)	0.028(1744)
<b>mod05</b>	21(11640)	0.25(4540)	0.06(5007)	0.036(1747)
<b>mod06*</b>	340(23193)	93(69322)	0.06(1637)	0.044(1637)
<b>mod07*</b>	703(31347)	141(101657)	0.14(5032)	0.098(2955)
<b>mod08*</b>	1593(42709)	267(157612)	0.87(20658)	0.15(5032)
<b>mod22*</b>	133(25620)	14.3(32106)	10.14(113548)	100(110445)
<b>mod23*</b>	52(17533)	20.6(45287)	75.64(230822)	8.9(58213)
<b>subst15*</b>	0.38(4)	0.38(4)	0.07(350)	0.060(98)
<b>syn02*</b>	0.0017(0)	0.0020(0)	0.14(0)	0.0035(0)
<b>syn11*</b>	0.0006(4)	0.0005(4)	0.024(0)	0.0098(0)
<b>syn12*</b>	0.0022(1)	0.0022(1)	0.13(0)	0.012(0)
<b>syn14*</b>	3.81(1461)	0.32(2351)	0.04(0)	0.035(1634)

Figure 1: Results for Matching Using Local Consistency

Problem	$S = 1$	$S = 2$	$S = 3$	$S = 1$ ( <b>flat</b> )
<b>mod01</b>	257(104268)	256(104268)	283(104268)	1712(670938)
<b>mod02</b>	334(90012)	359(90012)	330(90012)	1397(585402)
<b>mod03</b>	148(75288)	142(75288)	150(75324)	639(464658)
<b>mod04</b>	42(35985)	41(35985)	46(35985)	182(194905)
<b>mod05</b>	39(35110)	40(35110)	43(35110)	133(190530)
<b>mod06*</b>	577(27689)	602(27689)	443(32467)	593(125993)
<b>mod07*</b>	946(32338)	962(32338)	669(35410)	888(155391)
<b>mod08*</b>	1580(42193)	1719(42193)	1091(39057)	1258(175607)
<b>mod22*</b>	379(30758)	355(30758)	243(26814)	62(53035)
<b>mod23*</b>	92(18228)	91(18228)	69(14662)	26(26384)
<b>subst15*</b>	0.42(43)	0.44(43)	0.66(43)	0.42(43)
<b>syn02*</b>	0.013(0)	0.01(0)	0.015(0)	0.012(0)
<b>syn11*</b>	0.006(2)	0.019(26)	0.063(14)	0.006(2)
<b>syn12*</b>	0.01(0)	0.015(0)	0.031(0)	0.008(0)
<b>syn14*</b>	0.098(132)	0.12(126)	0.33(92)	0.0023(195)

somewhat worse than Algorithm 3.3 with learning of unrestricted lemmas. Since flattening of lemmas improves the performance of Algorithm 3.3 dramatically, we tried the same with the refining algorithm. The last column of Figure 8 shows that flattening has a big, negative impact on the refining algorithm.

## 9. FINDING OPTIMAL MATCHINGS

In this section we address the problem of finding optimal matchings. For the effectiveness of geometric resolution, it is important that a minimal matching is returned, in case more

than one exists. A minimal matching is a matching that uses the smallest possible set of assumptions. In terminology of DPLL, assumptions represent decision levels. The assumptions contributing to a conflict represent choice options, which will be replaced by other options during backtracking. In addition to being as few as possible, assumptions at a lower decision level should always be preferred over assumptions at a higher decision level. The reason for this is the fact that in other branches of the search tree, there is a risk that more assumptions will be used, and when assumptions are at a lower level, there is less room for this.

**Definition 9.1.** Let  $I$  be an interpretation. A weight function  $\alpha$  is a function that assigns finite subsets of natural numbers to the atoms of  $I$ .

Let  $A$  be a geometric literal. Let  $\Theta$  be a substitution such that  $A\Theta$  is in conflict with  $I$ . Referring to definition 1.3, we define  $\alpha(p_\lambda(x_1, \dots, x_n)\Theta, I) = \alpha(p_\mu(x_1\Theta, \dots, x_n\Theta))$ ,  $\alpha(x_1 \approx x_2)\Theta, I = \{\}$ , and  $\alpha(\#_f x)\Theta, I = \alpha(\#_t x\Theta)$ .

**Definition 9.2.** Let  $I$  and  $\phi = A_1, \dots, A_p \mid B_1, \dots, B_q$  together form an instance of the matching problem (Definition 1.5). Assume that  $\Theta$  is a solution. The *weight* of  $\Theta$ , for which we write  $\alpha(I, \phi, \Theta)$ , is defined as

$$\bigcup \left\{ \begin{array}{l} \{ \alpha(A_i\Theta, I) \mid 1 \leq i \leq p \} \\ \{ \alpha(C, I) \mid 1 \leq j \leq q, C \in E(B_j, \Theta), \text{ and } C \text{ conflicts } I \} \end{array} \right\}$$

Solving optimal matching means: First establish if  $(I, \phi)$  has a solution. If it has, then find a solution  $\Theta$  for which  $\alpha(I, \phi, \Theta)$  is multiset minimal.

One could try to impose further selection criteria that are harder to explain and whose advantage is less evident.

Solving the minimal matching problem is non-trivial, because the number of possible solutions can be very large. The straightforward solution is to use some efficient algorithm (e.g. the one in this paper) that enumerates all solutions, and keeps the best solution. Unfortunately, this approach is completely impractical because some instances have a very high number of solutions. One frequently encounters instances with  $> 10^9$  solutions.

In order to find a minimal solution without enumerating all solutions, one can use any algorithm that stops on the first solution in the following way: The first call is used to find out whether a solution exists. If not, then we are done. Otherwise, the algorithm is called again with its input restricted in such a way that it has to find a better solution than the previous. One can continue doing this, until all possibilities to improve the solution have been exhausted. It can be shown that the number of calls needed to obtain an optimal solution is linear in the size of the assumption set of solution. In this way, it can be avoided that all solutions have to be enumerated.

**Definition 9.3.** Let  $I$  be an interpretation that is equipped with a weight function  $\alpha$ . Let  $\phi = A_1, \dots, A_p \mid B_1, \dots, B_q$  be a geometric formula. Let  $\alpha$  be a fixed set of natural numbers. We define the  $\alpha$ -restricted translation  $(\Sigma^+, \Sigma^-)$  of  $(I, \phi)$  as follows:

- For every  $A_i$ , let  $\bar{v}_i$  be the variables of  $A_i$ . Then  $\Sigma^+$  contains the clause

$$\{\bar{v}_i/\bar{v}_i\Theta \mid A_i\Theta \text{ is in conflict with } I \text{ and } \alpha(A_i\Theta, I) \subseteq \alpha\}.$$

- For each  $B_j$ , let  $\bar{w}_j$  denote the variables of  $B_j$ . For every  $\Theta$  that makes  $B_j\Theta$  true in  $I$ ,  $\Sigma^-$  contains the substlet  $\bar{w}_j/(\bar{w}_j\Theta)$ . In addition, if there exists a  $C \in E(B_j, \Theta)$  that is in conflict with  $I$  and for which  $\alpha(C, \Theta) \not\subseteq \alpha$ , then  $\Sigma^-$  contains the substlet  $\bar{w}_j/(\bar{w}_j\Theta)$ .

The  $\alpha$ -restricted translation ensures that only conflicts involving atoms  $C$  with  $\alpha(C) \subseteq \alpha$  are considered, and (independently of  $\alpha$ ), that no  $B_j$  is made true. The translation of Definition 2.4 can be viewed as a special case of  $\alpha$ -restricted translation with  $\alpha = \mathbb{N}$ .

**Theorem 9.4.** *Let  $(\Sigma^+, \Sigma^-)$  be obtained by  $\alpha$ -restricted translation of  $(I, \phi)$ . For every substitution  $\Theta$ ,  $\Theta$  is a solution of  $(\Sigma^+, \Sigma^-)$  iff  $\Theta$  is a solution of  $(I, \phi)$ , and it has  $\alpha(I, \phi, \Theta) \subseteq \alpha$ .*

Using  $\alpha$ -restricted translation, we can define the **optimal** matching algorithm:

**Algorithm 9.5.** Let **solve** $(\Sigma^+, \Sigma^-)$  be a function that returns some solution of  $(\Sigma^+, \Sigma^-)$  if it has a solution, and  $\perp$  otherwise.

We define the algorithm **optimal** $(I, \phi)$  that returns an optimal solution of  $(I, \phi)$  if one exists and  $\perp$  otherwise.

- (1) Let  $(\Sigma^+, \Sigma^-)$  be the GCSP obtained by the translation of Definition 2.4. If  $\Sigma^+$  contains an empty clause, then return  $\perp$ . If  $\Sigma^-$  contains a propositional blocking, then return  $\perp$ . Otherwise, remove unit blockings from  $(\Sigma^+, \Sigma^-)$ . If this results in  $\Sigma^+$  containing an empty clause, then return  $\perp$ .
- (2) Let  $\Theta = \text{solve}(\Sigma^+, \Sigma^-)$ . If  $\Theta = \perp$ , then **return**  $\perp$ .
- (3) Let  $\alpha = \alpha(I, \phi, \Theta)$ , and let  $k := \sup(\alpha)$ .
- (4) As long as  $k \neq 0$ , do the following:
  - Set  $k = k - 1$ . If  $k \in \alpha$ , then do
    - Let  $\alpha' = (\alpha \setminus \{k\}) \cup \{0, 1, 2, \dots, k-1\}$ .
    - Let  $(\Sigma^+, \Sigma^-)$  be the  $\alpha'$ -restricted translation of  $(I, \phi)$ .
    - If  $\Sigma^+$  contains an empty clause or  $\Sigma^-$  contains a propositional blocking, then skip the rest of the loop. Otherwise, remove the unit blockings from  $(\Sigma^+, \Sigma^-)$ . If this results in  $\Sigma^+$  containing the empty clause, then skip the rest of the loop.
    - Let  $\Theta' = \text{solve}(\Sigma^+, \Sigma^-)$ . If  $\Theta' \neq \perp$ , then set  $\Theta = \Theta'$  and  $\alpha = \alpha(I, \phi, \Theta)$ .
- (5) Now  $\Theta$  is an optimal solution, so we can **return**  $\Theta$ .

Algorithm **optimal** first solves  $(I, \phi)$  without restriction. If this results in a solution  $\Theta$ , it checks for each  $k \in \alpha(I, \phi, \Theta)$  if  $k$  can be removed. The invariant of the main loop is: There exists no  $k' \geq k$  that occurs in  $\alpha(I, \phi, \Theta)$ , and no  $\Theta'$  that is a solution of  $(I, \phi)$  with  $k' \notin \alpha(I, \phi, \Theta')$ . In addition, the invariant  $\alpha = \alpha(I, \phi, \Theta)$  is maintained.

**Example 9.6.** Assume that in example 1.6, the atoms have weights as follows:

$$\begin{aligned} \alpha(P_t(c_0, c_0)) &= \{1\}, & \alpha(P_e(c_0, c_1)) &= \{2\}, & \alpha(P_t(c_1, c_1)) &= \{3\}, \\ \alpha(P_e(c_1, c_2)) &= \{4\}, & \alpha(Q_t(c_2, c_0)) &= \{5\}. \end{aligned}$$

We have  $\alpha(I, \phi_1, \Theta_1) = \{1\}$ ,  $\alpha(I, \phi_1, \Theta_2) = \{1, 2\}$ , and  $\alpha(I, \phi_1, \Theta_3) = \{2, 3\}$ . If  $\Theta_3$  is the first solution generated, **solve** will construct the  $\{1, 2\}$ -restricted translation of  $(I, \phi_1)$ , which equals

$$\frac{\begin{array}{l} (X, Y) / (c_0, c_0) \mid (c_0, c_1) \\ (Y, Z) / (c_0, c_0) \mid (c_0, c_1) \end{array}}{(X, Z) / (c_0, c_2)}$$

If the next solution found is  $\Theta_2$ , then **solve** will construct the  $\{1\}$ -restricted translation

$$\frac{\begin{array}{l} (X, Y) / (c_0, c_0) \\ (Y, Z) / (c_0, c_0) \end{array}}{(X, Z) / (c_0, c_2)}$$



whose only solution is  $\Theta_1$ .

## 10. FILTERING BY LOCAL CONSISTENCY CHECKING

Filtering is any procedure that simplifies or possibly rejects a GCSP before the main algorithm is called. In **Geo**, we have used filtering based on local consistency checking. In earlier versions, this was effective because very often, filtering rejects a GCSP without calling the main algorithm. Since the algorithms that we present in this paper, are much more efficient, this is not certain anymore. We still present the local consistency checking procedure, because it is easy to implement using refinement stacks, and it may be still an effective tool for filtering out easy instances.

Local consistency checking (see [8, 11, 14]) is a pre-check that comes in many variations. Local consistency checking is the following procedure: For every clause  $c = \{s_1, \dots, s_n\} \in \Sigma^+$ , check, for all sets of clauses  $C$  with size  $S \geq 1$ , if  $\{s_i\} \cup C$  has a solution. If not, then remove  $s_i$  from  $c$ . Keep on doing this, until no further changes are possible or a clause has become empty. Local consistency checking rejects a large percentage of GCSP instances a priori, and usually decreases the size of the clauses involved by a factor two or three.

In [8] (Chapter 3), local consistency checking is defined using subsets of variables (instead of clauses). Using subsets of two variables is called *arc consistency checking*, while considering subsets of three variables is called *path consistency checking*. In general, using bigger subsets is a more effective precheck, but also more costly because it gets closer to the original problem.

As discussed in Section 5, we had assumed in [5] that filtering is so effective, that one can base the complete search algorithm on it. Although this is possible in theory, the resulting algorithm turned out not competitive.

Since the local consistency checks the substlets in a single clause  $c$  against sets of clauses  $C \subseteq \Sigma^+$ , we define the size  $S$  of a local consistency check as  $S = \|C\|$ . When performing a local consistency check up to size  $S$ , one has to generate subsets up to size  $S + 1$ , and generate their solutions. If  $\|\Sigma^+\| = n$ , the total number of such subsets equals  $\binom{n}{S+1}$ , which grows very quickly for realistic  $n$ . The problem can be decreased by not generating all subsets, but only generate subsets whose clauses share variables, or have variables that co-occur in a blocking.

**Definition 10.1.** Let  $c, c'$  be clauses. We write  $c \sim c'$  if either  $c$  and  $c'$  share a variable, or there exist connected (Definition 3.2) variables  $v$  and  $v'$ , s.t.  $v$  occurs in  $c$  and  $v'$  occurs in  $c'$ .

It is sufficient to generate subsets that are connected, because consideration of subsets that are not connected will not lead to the removal of more substlets. We always assume that solutions are non-redundant, i.e. do not contain irrelevant assignments.

**Lemma 10.2.** Let  $(\Sigma^+, \Sigma^-)$  be a GCSP. Let  $C \subseteq \Sigma^+$ . If  $C$  can be written as  $C_1 \cup C_2$ , s.t. there exist no  $c_1 \in C_1$  and no  $c_2 \in C_2$  with  $c_1 \sim c_2$ , then for every two substitutions  $\Theta_1, \Theta_2$ , s.t.  $\Theta_1$  is a solution of  $(C_1, \Sigma^-)$  and  $\Theta_2$  is a solution of  $(C_2, \Sigma^-)$ ,  $\Theta_1 \cup \Theta_2$  is a solution of  $(C_1 \cup C_2, \Sigma^-)$ .

Lemma 10.2 guarantees that it is not needed to attempt to remove substlets from clauses in  $C_1 \cup C_2$ , after  $C_1$  and  $C_2$  have been checked. If some substlet  $s$  in  $C_1$  occur some solution of  $C_1$ , and  $C_2$  has a solution, then  $s$  will occur in the combined solution.

We will now show that instead of ignoring disconnected subsets, one can also ignore subsets that are connected only through a single clause:

**Lemma 10.3.** *Let  $(\Sigma^+, \Sigma^-)$  be a GCSP. Assume that  $C_1, C_2 \subseteq \Sigma^+$ , and  $c \in \Sigma^+$ . Assume that for every pair of variables  $v_1$  occurring in a clause of  $C_1$ , and  $v_2$  occurring in a clause of  $C_2$ , if either  $v_1 = v_2$  or  $v_1$  and  $v_2$  are connected, then  $v_1, v_2$  occur in  $c$ .*

*Then the following holds: If  $\Theta_1$  is a solution of  $(C_1 \cup \{c\}, \Sigma^-)$  and  $\Theta_2$  is a solution of  $(C_2 \cup \{c\}, \Sigma^-)$ , s.t.  $\Theta_1, \Theta_2$  agree on the variables occurring in  $c$ , then  $\Theta_1 \cup \Theta_2$  is a solution of  $(C_1 \cup C_2 \cup \{c\}, \Sigma^-)$ .*

*Proof.* Assume that  $\Theta_1, \Theta_2, C_1, C_2, c$  fulfill the conditions of the lemma. By non-redundancy,  $\Theta_1$  does not contain assignments to variables not occurring in  $c$  or  $C_1$ . Similarly,  $\Theta_2$  does not contain assignments to variables not occurring in  $c$  or  $C_2$ . If  $\Theta_1, \Theta_2$  share a variable  $v$ , then this variable must occur in  $c$ , which implies that  $v_{\Theta_1} = v_{\Theta_2}$ . As a consequence,  $\Theta_1$  and  $\Theta_2$  can be merged into a single substitution  $\Theta = \Theta_1 \cup \Theta_2$ , which has  $\Theta \models C_1 \cup C_2 \cup \{c\}$ .

If there would be a blocking  $\sigma \in \Sigma^-$ , s.t.  $\Theta \models \sigma$ , then we still have  $\Theta_1 \not\models \sigma$  and  $\Theta_2 \not\models \sigma$ . This implies that there are variables  $v_1$  in  $C_1 \setminus \{c\}$  and  $v_2$  occurring in  $C_2 \setminus \{c\}$ , which occur together in  $\sigma$ . But this contradicts the fact that  $v_1$  and  $v_2$  cannot be connected.  $\square$

As above, if some substlet  $s \in C_1$  is used in a solution of  $C_1 \cup \{c\}$ , and  $\{c\} \cup C_2$  has a solution, then the solutions can be combined into a single solution that uses  $s$ . If some substlet  $s$  of  $c$  occurs in a solution of  $C_1 \cup \{c\}$  and in a solution of  $\{c\} \cup C_2$ , then the solutions can be combined into a single solution that still uses  $s$ .

This implies that, if one uses a local consistency checker that gives preference to small subsets, one can ignore subsets that do not contain 'cycles'. If there exist  $c_1, c_2, c_3 \in C$ , s.t.  $c_1, c_3$  are not connected, and every path from  $c_1$  to  $c_3$  has to pass through  $c_2$ , then  $C$  can be ignored. This gives rise to the following definition:

**Definition 10.4.** Let  $(c_1, \dots, c_{S+1})$  with  $S \geq 1$  be a sequence of clauses. We call  $(c_1, \dots, c_{S+1})$  a *circle* if for every  $i$  ( $i \leq S$ ), we have  $c_i \sim c_{i+1}$ , and in addition we have  $c_{S+1} \sim c_1$ .

If  $\overline{C}$  is a refinement stack, we call a sequence of indices  $(i_1, \dots, i_{S+1})$  a *circle* if each  $\alpha_{i_j}(\overline{C})$  is true, and  $(d_{i_1}, \dots, d_{i_{S+1}})$  is a circle.

The local consistency checker checks only circles. Generation of circles in  $\Sigma^+$  is easier to implement than generation of all connected subsets, especially if one wants to avoid generating the same subset in different ways. In addition, it is more efficient because there are less circles than connected subsets. The discussion above suggests that generating circles is sufficient to obtain a complete check. We have believed for some time that this is true in general, but we will show below that it is false.

**Algorithm 10.5.** Let  $S \geq 1$  be a natural number. Let  $\Theta$  be a substitution. Let  $\overline{C}$  be a refinement stack.

A call to **local** $(s, \Theta, (k_1, \dots, k_{S+1}), \overline{C})$  constructs a refinement of  $\overline{C}$  by removing the substlets that do not occur in any solution of the subset of  $\Sigma^+$  of size  $S + 1$ .

It returns  $\perp$  if it establishes that  $\Theta$  cannot be extended into a solution of  $\overline{C}$ . Initially  $s = k_1 = \dots = k_{S+1} = 1$ .

**SUBST:** As long as  $s \leq \|\Theta\|$ , let  $v/x$  be the  $s$ -th assignment in  $\Theta$ .

(1) For every blocking  $\sigma \in \Sigma^-$  involving  $v$ , check if  $\Theta \models \sigma$ . If yes, then return  $\perp$ .

- (2) For every  $(c_i \Rightarrow d_i) \in \overline{C}$  which has  $\alpha_i(\overline{C})$  true and which contains  $v$ , let  $d'$  be the set of substlets in  $d_i$  that are consistent with  $\Theta$ . If  $d' = \emptyset$ , then return  $\perp$ . Otherwise, if  $\emptyset \subset d' \subset d$ , append  $(c_i \Rightarrow d')$  to  $\overline{C}$ .

**CLAUSES1:** As long as  $k_1 < \|\overline{C}\|$  do the following:

- (1) If  $\alpha_{k_1}(\overline{C})$  is true, and the  $k_1$ -refinement  $(c_{k_1} \Rightarrow d_{k_1})$  contains a variable  $v$ , s.t. all substlets  $(\overline{v}/\overline{x}) \in d_{k_1}$  agree on the assignment to  $v$ , then let  $x$  be the agreed value. Append  $v/x$  to  $\Theta$ .
- (2) Set  $k_1 := k_1 + 1$ .

If  $s \leq \|\Theta\|$ , then restart at **SUBST**. (This means that  $\Theta$  was extended in the previous step.)

**CLAUSESN:** As long as there is an  $i$  with  $2 \leq i \leq S + 1$ , s.t.  $k_i \leq \|\overline{C}\|$ , pick the smallest such  $i$ . If  $\alpha_{k_i}(\overline{C})$  holds, then

- (1) Enumerate all circles  $(\lambda_1, \dots, \lambda_i)$  of size  $i$  starting at  $\lambda_1 = k_i$ . For each such circle  $(\lambda_1, \dots, \lambda_i)$ , let  $I = \{d_{\lambda_1}, \dots, d_{\lambda_i}\}$ . Call **refine** $(I, \Theta, \overline{C})$ . If the result is  $\perp$ , then return  $\perp$ . If after the call, we have  $\|\overline{C}\| > k_1$ , then restart at **CLAUSES1**.

Let  $\overline{C}$  be a refinement stack. Let  $k = \|\overline{C}\|$ . Let  $I$  be a subset of  $\{1, \dots, k\}$ , s.t. for every  $i \in I$ ,  $\alpha_i(\overline{C})$  holds. Algorithm **refine** $(I, \Theta, \overline{C})$  is defined as follows:

- (1) Initialize a map  $U$  with domain  $I$  by setting  $U(i) = \emptyset$ , for each  $i \in I$ . Eventually,  $U$  will map each  $i \in I$  to the set of substlets in  $d_i$ , that can occur in a solution  $\Theta'$  of  $\{d_i \mid i \in I\}$  extending  $\Theta$ .
- (2) Enumerate all maps  $S$  with domain  $I$  that map each  $i \in I$  to a substlet  $S(i)$  in  $d_i$ , and that have the following properties: No  $S(i)$  conflicts  $\Theta$ , no  $S(i), S(i')$  are in conflict with each other.  $\Theta \cup \{S(i) \mid i \in I\}$  does not imply a blocking  $\sigma \in \Sigma^-$ . For each of the generated mappings  $S$ , for each  $i \in I$ , set  $U(i) = U(i) \cup \{S(i)\}$ .
- (3) For every  $i \in I$ , for which  $U(i) \neq d_i$ , add the refinement  $(c_i \Rightarrow U(i))$  to  $\overline{C}$ .

The local consistency checker gives priority to checking against the substitution. After checking for conflicts against the substitution, Algorithm 10.5 generates circles of size up to  $S + 1$ , and checks for each of the substlets occurring in the clauses of such a circle, whether it can occur in a solution. Substlets that do not occur in a solution are refined away. Preference is given to small circles. This means that circles of size  $i + 1$  will be checked only after all circles up to size  $i$  have been checked.

We will discuss (and disprove) the conjecture mentioned above, that it is sufficient to check circles, when preference is given to smaller subsets. More precisely: If for a given subset  $C \subseteq \Sigma^+$ , all its subcircles have been checked, then  $C$  needs to be checked only if it is a circle by itself. We formally define what 'has been checked' means:

**Definition 10.6.** Let  $(\Sigma^+, \Sigma^-)$  be a GCSP. Let  $\Theta$  be a substitution. Let  $C$  be a subset of clauses of  $\Sigma^+$ . We write  $\Phi(C)$  for the following property: For every clause  $c \in C$ , for every substlet  $s \in c$ , there is a solution  $\Theta$  of  $(C, \Sigma^-)$ , s.t.  $\Theta \models s$ .

Algorithm 10.5 tries to establish  $\Phi(C)$  for every subset  $C$  of size  $i$  up to  $S + 1$ . It assumes that when  $\Phi(C)$  holds for circles with size smaller than  $\|\overline{C}\|$ , and  $C$  is not a circle, then  $\Phi(C)$  automatically holds. We have believed for some time that this assumption is true, because Lemma 10.2 and Lemma 10.3 provide evidence for it, and it simplifies Algorithm 10.5. Unfortunately, the property fails at  $S = 4$ , for circles of size 5.

**Conjecture 10.7.** Let  $(\Sigma^+, \Sigma^-)$  be a GCSP. Assume that every strict subset  $C' \subset \{c_1, \dots, c_{S+1}\}$  that can be arranged into a circle  $c'_1, \dots, c'_{S'+1}$  has property  $\Phi(C')$ . Then if  $C$  cannot be arranged into a circle,  $C$  has the property  $\Phi(C)$ .

We prove Conjecture 10.7 for  $S \leq 4$ , and provide a counter example for  $S = 5$ .

*Proof.* •  $S = 1$  follows from Lemma 10.2.

- In order to prove  $S = 2$ , assume that  $c_1, c_2, c_3$  are clauses that do not form a circle. Without loss of generality, we may assume that  $c_1 \not\sim c_3$ . If we also have  $c_1 \not\sim c_2$ , then  $\{c_1, c_2, c_3\}$  can be partitioned into  $\{c_1\}, \{c_2, c_3\}$ , so that Lemma 10.2 can be applied. If we have  $c_1 \sim c_2$ , we can apply Lemma 10.3 with  $C_1 = \{c_1\}$ ,  $c = c_2$ ,  $C_2 = \{c_3\}$ .
- We prove  $S = 3$ . We use the fact that Conjecture 10.7 holds for  $S < 3$ . Let  $c_1, c_2, c_3, c_4 \in \Sigma^+$ . If  $\{c_1, c_2, c_3, c_4\}$  be be partitioned into two disjoints sets, we can apply Lemma 10.2, and we are done. Otherwise, if  $\{c_1, c_2, c_3, c_4\}$  cannot be partitioned into disconnected sets, there are two possibilities:
  - The clauses form a line  $c_1 \sim c_2 \sim c_3 \sim c_4$ . If  $c_1 \sim c_4$ , then  $(c_1, c_2, c_3, c_4)$  is a circle, so that Conjecture 10.7 holds trivially. Otherwise, we can still have  $c_1 \sim c_3$  or  $c_2 \sim c_4$ . If he have both, then  $(c_1, c_3, c_4, c_2)$  is a circle, so that Conjecture 10.7 again holds trivially. If  $c_1 \not\sim c_3$ , we can apply Lemma 10.3 with  $C_1 = \{c_1\}$ ,  $c = c_2$ , and  $C_2 = \{c_3, c_4\}$ . Similarly, if  $c_2 \not\sim c_4$ , we can apply Lemma 10.3 with  $C_1 = \{c_1, c_2\}$ ,  $c = c_3$ , and  $C_2 = \{c_4\}$ .
  - The clauses form a kind of star with  $c_1$  in the center:  $c_1 \sim c_2$ ,  $c_1 \sim c_3$ ,  $c_1 \sim c_4$ . For  $c_2$ , if  $c_2 \sim c_3$ , nor  $c_2 \sim c_4$ , we can apply Lemma 10.3 with  $C_1 = \{c_2\}$ ,  $c = c_1$ ,  $C_2 = \{c_3, c_4\}$ . If we have both of  $c_2 \sim c_3$  and  $c_2 \sim c_4$ , then  $(c_2, c_3, c_1, c_4)$  is a circle. In the remaining case, we may assume without loss of generality that  $c_2 \sim c_3$ , but also  $c_2 \not\sim c_4$ . This means that we have  $c_2 \sim c_3, c_2 \not\sim c_4$ . If  $c_4 \sim c_3$ , then  $(c_1, c_2, c_3, c_4)$  is a circle. If  $c_4 \not\sim c_3$ , then we can apply Lemma 10.3 with  $C_1 = \{c_2, c_3\}$ ,  $c = c_1$ ,  $C_2 = \{c_4\}$ .

□

We give a counter example for  $S = 4$ .

**Example 10.8.** Consider the following GCSP, which has no blockings, and the following clauses:

$$\begin{aligned}
 (c_1) \quad & (X_1, X_2, X_3) / (0, 0, 0) \mid (0, 1, 1) \mid (1, 1, 0) \mid (1, 0, 1) \\
 (c_2) \quad & (X_1, Y_1) / (0, 0) \mid (1, 1) \\
 (c_3) \quad & (X_2, Y_2) / (0, 0) \mid (1, 1) \\
 (c_4) \quad & (X_3, Y_3) / (0, 0) \mid (1, 1) \\
 (c_5) \quad & (Y_1, Y_2, Y_3) / (1, 0, 0) \mid (0, 1, 0) \mid (0, 0, 1) \mid (1, 1, 1)
 \end{aligned}$$

We have  $c_1 \sim c_2$ ,  $c_1 \sim c_3$ ,  $c_1 \sim c_4$ , and  $c_2 \sim c_5$ ,  $c_3 \sim c_5$ ,  $c_4 \sim c_5$ . There are no other connections. The example can be understood as follows: Clause  $c_2$  requires that  $X_1\Theta = Y_1\Theta$ . Similarly,  $c_3$  requires that  $X_2\Theta = Y_2\Theta$ , and  $c_4$  requires that  $X_3\Theta = Y_3\Theta$ . Clause  $c_1$  requires that  $X_1\Theta + X_2\Theta + X_3\Theta$  is even, while  $c_5$  requires that  $Y_1\Theta + Y_2\Theta + Y_3\Theta$  is odd. Since the sums must be equal, and cannot be odd and even at the same time,  $(\{c_1, c_2, c_3, c_4, c_5\}, \{\})$  has no solution.

Ignoring direction and starting point, there are three circles of size 4 :

$$(c_1, c_2, c_5, c_3), (c_1, c_2, c_5, c_4), (c_1, c_3, c_5, c_4).$$

Since the circles are symmetric, we show that every substlet occurring in  $\{c_1, c_2, c_5, c_3\}$  can occur in a solution. One can pick the instance of  $c_1$  and  $c_5$  in such a way that they agree on  $X_1/Y_1$  and  $X_2/Y_2$ . They will disagree on  $X_3/Z_3$ , but because  $c_4$  is not considered, this is no problem. After that, the instances to  $c_2$  and  $c_3$  are fixed. It is easily checked that  $c_1, c_2, c_3, c_4, c_5$  cannot be arranged into circle.

Example 10.8 contains a GCSP that would not be refined by Algorithm 10.5 with  $S = 4$ , despite the fact that it has no solution. We will refrain from trying to make Algorithm 10.5 complete, because we believe that it is not worth the effort. Experiments suggest that using Algorithm 10.5 becomes too costly already at  $S \geq 3$ . Implementing a more elaborate check at  $S \geq 3$ , would make Algorithm 10.5 even more costly, and harder to implement, without much hope for improvement.

It should be noted that even when Algorithm 10.5 is used as precheck, it still needs to be restorable, because it may be called by Algorithm 9.5, which will turn on and off different substlets, based on  $\alpha$ . Only the first call need not be restorable.

## 11. CONCLUSIONS

The problem of matching a geometric formula into an interpretation is still the most costly part of our implementations of geometric resolution. In order to improve the situation, we gave a translation of the matching problem into GCSP, and we provided efficient algorithms for GCSP. Two of the algorithms solve the GCSP directly, and two of the algorithms solve the GCSP by translation to SAT. Currently, translation to SAT shows better performance, but this may be due to the fact that our current implementation is not optimal, because it uses datastructures from an earlier matching algorithm. Independent of the relative performances, we can conclude that the speed of **Geo** can be improved by a factor  $\approx 1000$ .

The fact that the clause refining algorithm turned out not competitive, shows that algorithms that appear to be good in theory, are not necessarily good in practice. In general, it is difficult to predict what will be the effect of a modification of an algorithm. A seemingly small change may have a large impact on performance.

One might argue that a calculus that uses an NP-complete problem as its basic operation is not viable, but there is room for interpretation: The complexity of the matching problem is caused by the fact that as result of flattening, geometric formulas and interpretations have DAG-structure instead of tree-structure. This increased expressiveness means that a geometric formula possibly represents exponentially many formulas with tree-structure. This may very well result in shorter proofs. Only experiments can determine which of the two effects will be stronger.

## 12. FUNDING

We gratefully acknowledge that this work was supported by the Polish National Science Center (Narodowe Centrum Nauki) under grant number DEC-2015/17/B/ST6/01898 (Application of Logic with Partial Functions).

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